Lecture IV

Last time, we defined $\dim_T$ and had shown $\dim_T(\mathbb{R}^n) \leq n$. To show $\dim_T(\mathbb{R}^n) = n$ we need to consider the question:

what properties does $\mathbb{R}^n$ have that $\mathbb{R}^{n-1}$ lacks?

A: 1. we can find n linearly independent vectors in $\mathbb{R}^n$. We cannot do this in $\mathbb{R}^{n-1}$.

we know $\dim_V(\mathbb{R}^{n-1}) = n-1$.

$\rightarrow$ $n-1$ is the number of vectors in any collection A s.t. $\text{span}(A) = \mathbb{R}^{n-1}$, A is linearly independent.

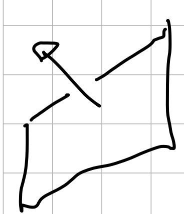
If we have a collection of $n-1$ vectors that are linearly independent.

then this collection spans $\mathbb{R}^{n-1}$,

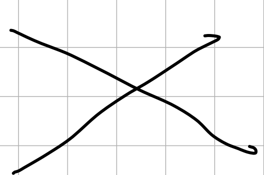
so adding any vector is adding a vector in the span.

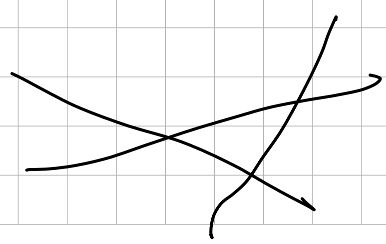
$\rightarrow$ linearly dependent.

Since each vector defines a perpendicular plane

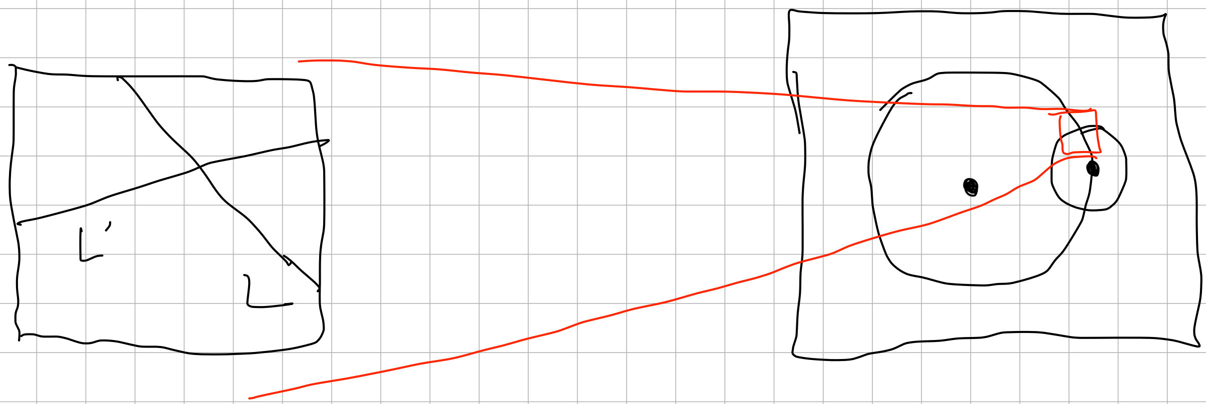


this means that "in general" the intersection of n-planes is non-empty in $\mathbb{R}^n$. Is empty in $\mathbb{R}^{n-1}$.





How do we use this for our notion of dimension?

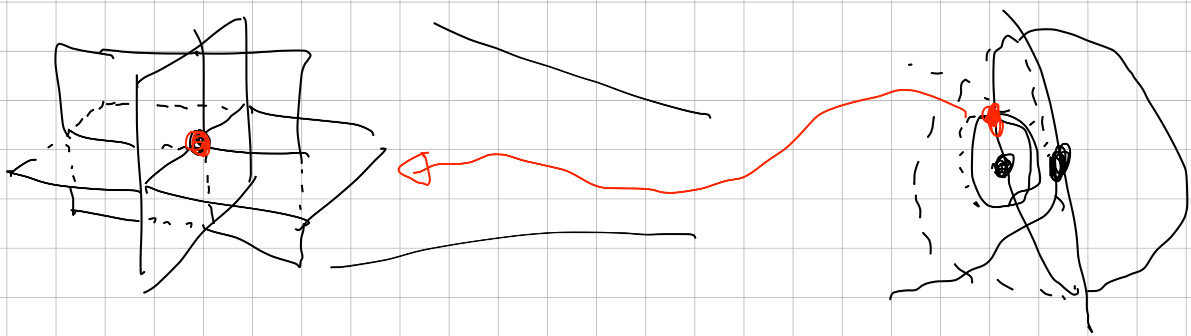


$\boxed{\text{Big Idea}}$ viewing hyperplanes as Boundaries

statements about intersections

$\rightsquigarrow$ are statements about dimension.

Consider $B_r(x)$ in $\mathbb{R}^3$.



If intersection is not empty, $\to \dim_T > -1$

$\to \dim_T(S^1) > 0$

$\to \dim_T(S^2) > 1$

which supports $\to \dim_T(\mathbb{R}^3) > 2$

This is the Intuition.

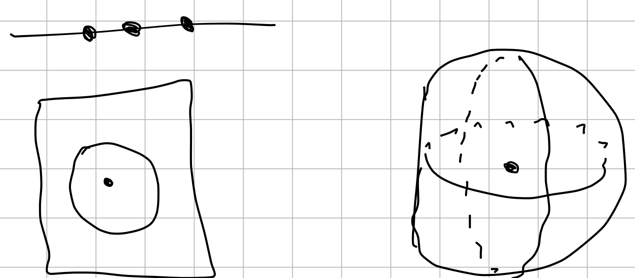
To formalize this intuition, to make it rigorous is actually quite hard. It requires a lot of theory and some surprising connections.

Problems to overcome

$\boxed{\#1.}$ Boundaries of open sets may not be planes. We need a notion that captures the behavior of boundaries of open sets.

$\boxed{\#2.}$ Once we have such a notion, we need to prove that the intersection properties that hold for planes, hold for these sets.

$\boxed{\text{How to tackle problem \#1}}$



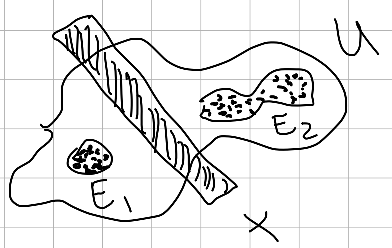
- $\partial B_r(p)$ carves up $\mathbb{R}^n$ into an inside $(B_r(p))$ and outside $(\overline{B_r(p)}^c)$.

- $\mathbb{R}^n \setminus \partial B_r(p)$ is disconnected, meaning
  $\mathbb{R}^n \setminus \partial B_r(p)$ is the union of two disjoint open sets.
- The boundary separates the inside from the outside.

Def: Let $X, E_1, E_2 \subseteq U \subseteq \mathbb{R}^n$ be disjoint.
X separates E_1 and E_2 in U iff
There exist two open sets V_1, V_2
such that

1. $V_1 \cap V_2 = \emptyset$
2. $U \setminus X \subseteq V_1 \cup V_2$
3. $E_1 \subseteq V_1$ and $E_2 \subseteq V_2$

$\boxed{\text{Ex}}$



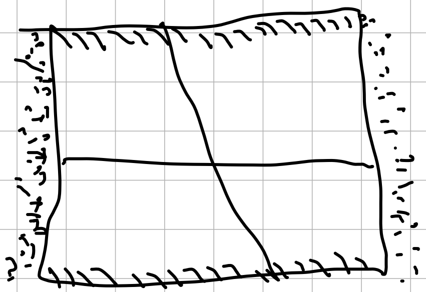
Intersection of separating sets still behave like planes.

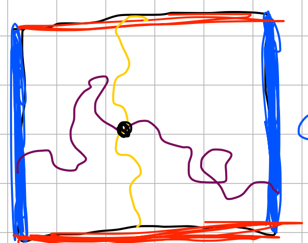
We generalize n planes intersecting in $\mathbb{R}^n$
as:

Theorem A: Let C_i, C_i' be opposite faces of $[0,1]^n$ the n-cube.

Let K_i be closed sets separating C_i, C_i' in $[0,1]^n$. Then

$$\bigcap_{i=1}^{n} K_i \neq \emptyset.$$





However, proving this theorem is not easy. It actually relies upon a very deep result called the Brouwer Fixed-Point theorem:

Theorem: (Brouwer Fixed-Point Theorem) Let $f: \overline{B_r(p)} \to \overline{B_r(p)}$ be a continuous function. Then $\exists x \in \overline{B_r(p)}$ s.t. $f(x) = x$.

This is one of the most famous results in topology (study of continuous functions)

Brouwer proved it in 1913 in a paper

in which he defines $\dim_T$ and proves $\dim_T \mathbb{R}^n = n$.

and I think that it is very interesting that in trying to prove that $\dim_T (\mathbb{R}^n) = n$ we need a result about properties of continuous maps on Balls.

A result about $(n-1)$-$\dim_T$ spaces

Theorem B:

If X has $\dim_T(X) \leq n-1$, and $\{C_i, C_i'\}_{i=1}^n$ are n pairs of closed sets in X, $C_i \cap C_i' = \emptyset$

$\exists B_i \subseteq X$ s.t. B_i separates C_i, C_i' s.t. $\bigcap_{i=1}^n B_i = \emptyset$.

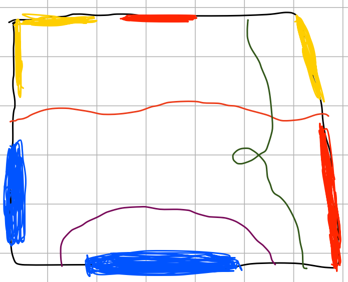
If we go back to our intuition with planes, (separate $\mathbb{R}^n$)

If $n=3$, $\dim_T(X) = n-1 = 2$

it is easy to see

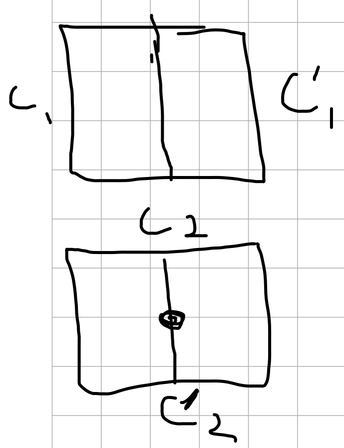
But this theorem

- applies to <u>all</u> X such that $\dim_T(X) = n-1$
- applies to <u>all</u> $\{C_i, C_i'\}_{i=1}^n$ pairs of disjoint closed sets.



No mutual intersection.

Idea:



Opt.

$\boxed{\text{Big Idea}}$ Suppose X, A s.t. $\dim_T(A) \le k$

and E_1, E_2 closed disjoint sets.

we need to be able to find a

closed set B separating E_1, E_2 in X

and $\dim_T(A \cap B) \le k-1$.

1. $A = X$, $k = n-1$, $E_1, E_2 = C_1, C_1'$

   get B_1 separating C_1, C_1' in X. $\dim_T(B_1) \le n-2$

2. $A = B_1$ $k = n-2$ $E_1 = C_2$ $E_2 = C_2'$

   get B_2 st. B_2 separates C_2, C_2'

   $\dim_T(B_2 \cap B_1) \le n-3$

n. $A = \bigcap_{j=1}^{n-1} B_j$, $k = 0$ $E_1 = C_n$, $E_2 = C_n'$

   get B_n separating C_n, C_n' in X and $\dim_T(B_n \cap \bigcap_{j=1}^{n-1} B_j) \le -1$

$\longrightarrow \bigcap^n B_j =$ emptyset.

Together Theorem A and Theorem B show that

$\dim_T([0,1]^n) > n-1$. Since $[0,1]^n \cong \mathbb{R}^n$ and

$\dim_T(\mathbb{R}^n) \leq n$, this gives $\dim_T(\mathbb{R}^n) = n$!

Therefore, $\dim_T$ satisfies all the properties we wanted:

1. applies to all subsets $X \subseteq \mathbb{R}^n$
2. $\dim_T(\mathbb{R}^n) = n$
3. If $f: X \to Y$ is a homeomorphism

   $\dim_T(X) = \dim_T(Y)$.

It gives the first general, rigorous theory of dimension. If you ever study topology, you will see it is a beautiful theory.

But it is not the only beautiful and general theory of dimension.

BACK TO THE DRAWING BOARD

To come up with a different formalization of dimension we need new intuitions to formalize. To get these intuitions, we could

a. consider a different class of functions
   functions of bounded derivatives
   $\to$ metric dimension theory.

b. continue to consider balls.

   We know we want dimension to depend upon local picture $\to$ balls,
   but we also need to account for local everywhere-

Define: For a set $E \subseteq \mathbb{R}^n$,

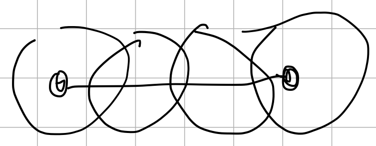
we define a closed (open) cover

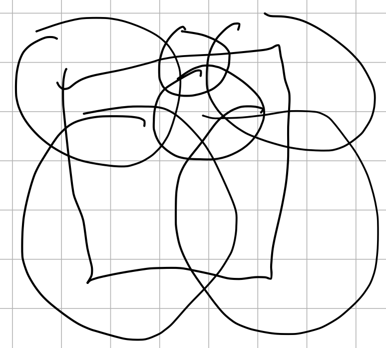
to be a collection of closed (open) balls

$\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ such that

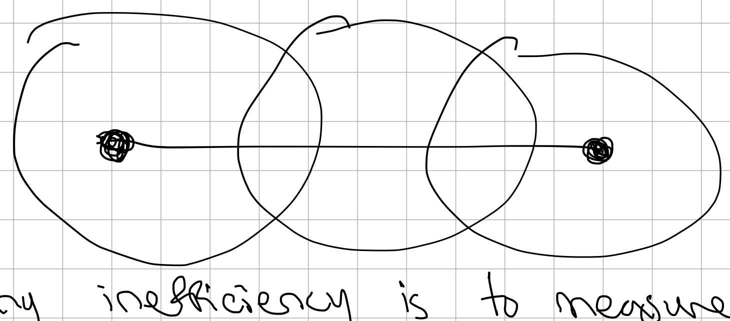
$$E \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i)$$

Q: what kind of properties can a cover have?





1. Number of balls used in a cover.
2. We might care about how *efficient* our cover is.



- one way of measuring *inefficiency* is to measure how much *redundancy* or *overlap* there is.

we can "measure" this overlap by

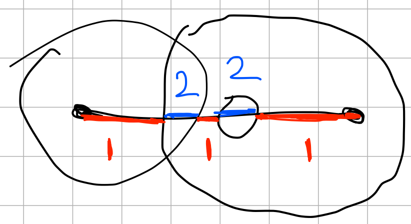
a. counting the amount of overlap

b. measuring the "size" of the set in which there is overlap.

Def: For a set $X \subset \mathbb{R}^n$ and a closed cover $\{K_i\}_{i=1}^N$ of X, for each $x \in X$, we define the covering index at x to be the number of K_i s.t. $x \in K_i$.

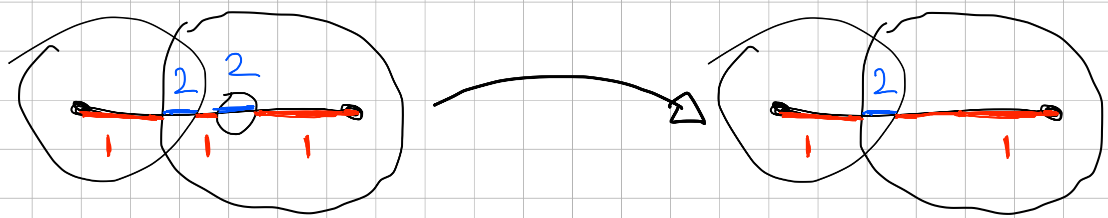
We define the covering index of X to be the maximum of the covering indexes at $x \in X$.

Ex



covering index of X is 2.

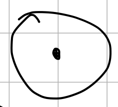
Q: How small can we make the covering index?

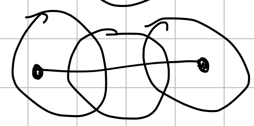


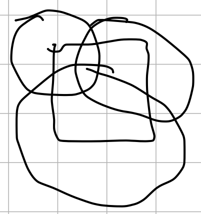
Sometimes we can refine our covering to obtain another covering. Note that the covering index decreases under this procedure.

Define: Given a cover $\{K_i\}_{i}$, we shall say that a different cover $\{D_i\}_i$ is a refinement of $\{K_i\}$ if each $D_i \subset K_j$ for some j.

Ex

a.  covering index 1

b.  covering index 2

c.  covering index 3

Based on this, we may make the following definition.

<u>Def:</u> For $X \subseteq \mathbb{R}^n$ we say

$\dim_{LC}(X) = n$ <u>iff</u> every closed cover has a refinement with cover index $\leq n+1$.

"The least inefficient a cover of an n-$\dim_{LC}$ set can be is $n+1$ overlapping balls."

With a new definition of dimension,

there are some natural questions:

Theory Questions:

1. Does $\dim_{LC}$ apply to all subsets of $\mathbb{R}^n$?

   Yes.

2. Is it invariant under homeomorphisms?

   could be (if we use open sets, not balls).

3. How does this definition of dimension compare with the $\dim_T$?

   Is it possible that there is a set $X \subseteq \mathbb{R}^n$ such that $\dim_{LC}(X) \neq \dim_T(X)$?

Take-Home Question

$\boxed{4.}$ What is $\dim_{LC}([0,1]^n)$?

You will need to use some of the theory we developed today.