Lecture # 1

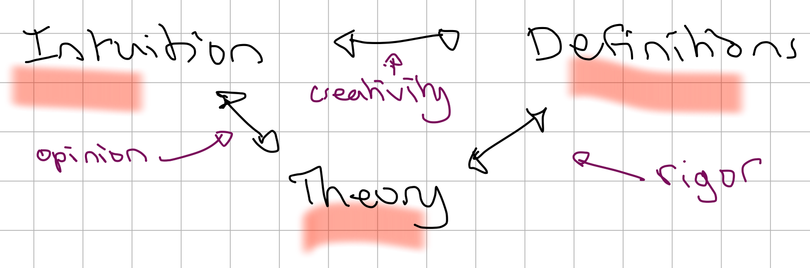
Introduction

"What is Dimension?"

- On the surface, this course is about exploring "Dimension"

- But there is a deeper purpose: to invite you into math as an open, creative exploration. Math as a study of mathematical concepts.

- I am a researcher, and I think of math as an interplay between 3 phases



- In this mini course, we are going to explore this interplay by focusing upon the question "What is Dimension?"

The Big Question

"What is dimension?"

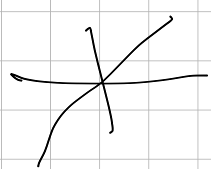
It can be hard to address such an abstract question.

So let's start by looking at some examples.

Ex 0. a point $\bullet$ $\{\vec{0}\}$

1. a line $\mathbb{R} = \{x : -\infty < x < \infty\}$

2. a plane  $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$

3. 3-space  $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$

Q: How many dimensions does each of these have?

A:

The BIG Question of this course (what is dimension?)

is really a question

"Why do we think a line is 1-dim,

a plane is 2-dim, and 3-space is 3-dim?"

So, why? what do you think?

A. The number of coordinates required to describe a point.

B. The number of "directions we can wiggle."

C. The number of "degrees of freedom."

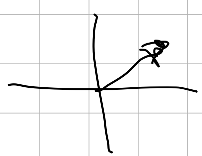
These explanations work in conversation and up until the late 19th Century, they were good enough for mathematics. But let's try to make them into an explicit definition.

Goal for today:

Formalize "number of directions we can wiggle."

One natural way to do this is using the language of Linear Algebra:

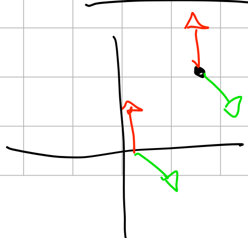
- We can think of "points" in $\mathbb{R}^n$ as "vectors."



direction and magnitude.

- we can think of "wiggling in a direction" as adding a non-zero vector to a point.

Geometrically



In coordinates

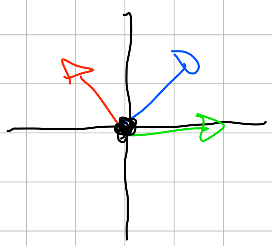
$$(x,y) + (v_1, v_2) = (x+v_1, y+v_2)$$

But, we don't really want to think of

"The number of directions you can wiggle"

as "The number of non-zero vectors you can add to a point."

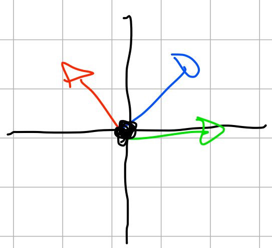
 1-dim

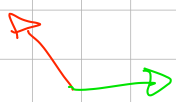
 2-dim

There are <u>lots of</u> vectors we can add. We do <u>not</u> think of these as wiggling in a <u>new</u> dimension

We need to come up with a sense in which

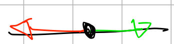
1.  These vectors are wiggling in the <u>same</u> direction

2. 

 are two different directions, but wiggling in  is a different direction, but

<u>Not</u> a new dimension.

One way to answer this is

$\boxed{1.}$  These different vectors are the "same direction" because they are on the <u>same line.</u>

$$\vec{v}_1 = c\,\vec{v}_2$$

$\boxed{2.}$ One sense in which $\vec{v}_3$ is a different vector, but NOT a new dimension is that $\vec{v}_3$ can be obtained by a "linear combination" of $\vec{v}_1, \vec{v}_2$

Geometrically

In coordinates

$\exists\, c_1, c_2 \in \mathbb{R}$ s.t.

$$\vec{v}_3 = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

This leads to the following definition.

Def: for a collection of vectors $A \subset \mathbb{R}^n$

span(A) = all the places one can wiggle to by wiggling in the directions in the collection A.

span(A) = all the vectors one can get to by taking linear combinations of vectors in A.

1. If $A = \emptyset$, we define $\text{span}(\emptyset) = \{\vec{0}\}$

2. If A is non-empty,

$$\text{span}(A) = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \text{ for } c_1, \dots, c_m \in \mathbb{R}\}$$

$$\forall m \in \mathbb{N}$$

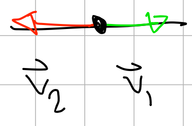
$\boxed{\text{Ex}}$

a. a point. $\{\vec{0}\}$ the only vector is $\vec{0}$

So the only possible non-empty collection $A = \{\vec{0}\}$

$\text{span}(A) = \{\vec{0}\}$.

b. a line.



For any non-zero $\vec{v}_1$,

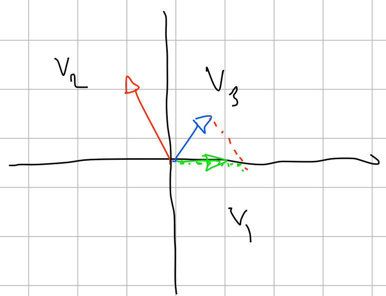
$\text{span}(\{\vec{v}_1\}) = \mathbb{R}$

For any $\vec{v}_2 \in \mathbb{R}$, $\text{span}(\{v_1, v_2\}) = \text{Span}(\{\vec{v}_1\})$

because $v_2 = c \cdot \vec{v}_1$

$$c_1 v_1 + c_2 v_2 = c_1 v_1 + c_2 (c v_1) = c_3 \vec{v}_1$$

c. a plane.



$\text{span}(\{\vec{v}_1\}) = x\text{-axis}$.

$\text{span}(\{\vec{v}_1, \vec{v}_2\}) = \mathbb{R}^2$.

$\text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}) = \mathbb{R}^2$.

Lessons: 1. if $\vec{v} \in \text{span}(A)$ then

$\text{span}(A \cup \{\vec{v}\}) = \text{span}(A)$.

2. If $\vec{v} \notin \text{span}(A)$, then

$\text{span}(A \cup \{\vec{v}\}) \supsetneq \text{span}(A)$.

$\to$ Idea: if $\text{span}(A)$ = all places you can wiggle to

then maybe "the number of directions you can wiggle" in X

is the smallest number of vectors in a

collection A s.t. $\text{span}(A) = X$.

Def: $\dim_r(X) :=$ the smallest number of vectors in a collection A s.t. $\operatorname{span}(A) = X$.

Ex if $X = \{\vec{0}\}$, $\operatorname{span}(\emptyset) = \{\vec{0}\} \to \dim_r(\{\vec{0}\}) = 0$.

if $X = \mathbb{R}^n$ for $n \geq 1$, this definition works, but we will introduce an equivalent, slightly more convenient definition.

We already saw that if $\vec{v} \in \operatorname{Span}(A)$,

then $\operatorname{span}(A \cup \{\vec{v}\}) = \operatorname{span}(A)$.

$\to \vec{v}$ is redundant.

if $A = \{\vec{v}_1, \dots, \vec{v}_n\}$

$v = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

Then $\vec{0} = -v + c_1\vec{v}_1 + \dots + c_n\vec{v}_n$.

Def: a collection of vectors $A \subseteq \mathbb{R}^n$

is called linearly dependent if

$\exists\, c_1, \dots, c_m \in \mathbb{R}$ s.t. not all $c_i = 0$ and

$\vec{0} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$.

It is called linearly independent if it is not linearly dependent.

Note: a collection of vectors $A = \{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly dependent if either

A) $\exists i$ s.t. $\vec{v}_i = \vec{0}$

B) $\exists i$ s.t. $\vec{v}_i \in \operatorname{span}(A \setminus \{\vec{v}_i\})$

"If A is linearly dependent, we can remove a vector without diminishing span."

Now we can state our formalization of

" the number of directions you can wiggle".

Def:

$\dim_{\checkmark}(X) :=$ number of vectors in a collection A s.t.

1) A is linearly independent

2) $\text{span}(A) = X$.

$\text{Span}(A) = X \to$ "all directions you can wiggle"

linearly independent $\to$ "number of different directions"

<u>Theory Questions:</u>

1. Is this definition well-posed?

Do all linearly independent, spanning collections have the same number of vectors? $\boxed{\text{YES.}}$ This is Linear algebra

2. How does this definition match our intuition?

Lets compute: $\boxed{\text{For } n > 0.}$

For $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}$
we expect $\mathbb{R}^n$ to be $\dim_V(\mathbb{R}^n) = n$

Consider $A = \{(1, 0, \dots, 0)\ (0, 1, \dots, 0)\ \dots\ (0, 0, \dots, 1)$

Since both span and linear independence have to do with linear combos,

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = c_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} + \cdots + c_n\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix} = \begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}$$

$\boxed{11.}$ Does A span $\mathbb{R}^n$?

$$\text{span}(A) = \left\{ (c_1, \cdots, c_n) : c_1, \ldots, c_n \in \mathbb{R} \right\} = \mathbb{R}^n. \quad \text{Yes!}$$

$\boxed{12.}$ Is A linearly independent?

Is there a linear combination (non-trivial)

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_n \quad \text{for} \quad c_1, \cdots, c_n \text{ nontrivial?}$$

$$\begin{aligned} c_1 + 0 + \cdots + 0 &= 0 \\ 0 + c_2 + \cdots + 0 &= 0 \\ &\ddots \\ c_n &= 0 \end{aligned}$$

$\longrightarrow c_i = 0$ for all $i = 1, \ldots, n$

A is linearly independent. Yes.

$\longrightarrow \dim_{\mathbb{R}}(\mathbb{R}^n) = n.$

So, this definition of "dimension" matches our intuition. Great! This is a beautiful definition and the language it comes from (Linear Algebra) is very powerful and important.

But, it has some big weaknesses, too.

This definition of dimension relies on the ability to "wiggle" at every point $x \in \mathbb{R}^n$.

This means adding points/vectors
and considering spans (linear combinations).
For all of this to be well-defined,
we need to be dealing with a
vector space.

Def: A vector space is a set V
of objects called vectors
on which we can define two operations.

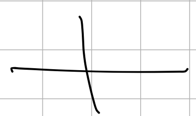
1. Addition
2. Scalar multiplication.

such that

1. For all $\vec{v}_1, \vec{v}_2 \in V$, $\vec{v}_1 + \vec{v}_2 \in V$
2. For all $c \in \mathbb{R}$, $\vec{v} \in V$, $c\vec{v} \in V$.

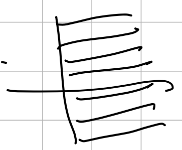
(and algebra works the way you think it does)

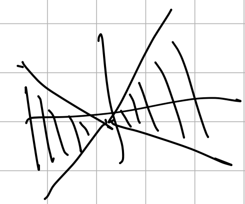
These are the operations we need
in order to talk about span & linear dependence.

$\boxed{\text{Ex}}$ 1. $\mathbb{R}^2$ is a vector space 

but what about smaller parts of $\mathbb{R}^2$?

2.  The x-axis is a vector space.

3.  is not closed under scalar multiplication

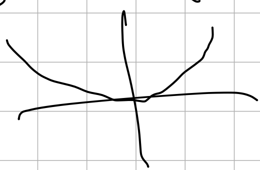
4.  is not closed under addition.

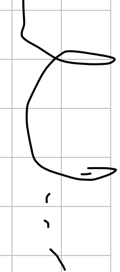
So, even though this definition of "dimension" is perfect for vector spaces, it cannot be applied (as it is) to non-vector spaces.

If we want a definition of dimension that applies to all subsets of $\mathbb{R}^n$ we need to be a little clever.

Take home Question

consider the following examples.

1. $\{(x,y) \mid y = x^2\}$ 

2. $\{(x,y,z) \mid (x,y,z) = (\cos(t), \sin(t), t)$ for $t \in \mathbb{R}\}$ 

- What dimension should these examples be?
- How might we use or modify $\dim_V$ to apply to these examples?