WHAT IS DIMENSION? DAY 2

SEAN MCCURDY

1. Day 2 Notes

1.1. Descriptions of sets.

Definition 1.1. Let $n \in \mathbb{N}$.

(1) A set $U \subset \mathbb{R}^n$ is called *open* if for all $x \in U$ there is a ball $B_r(x) \subset U$.
(2) A set $K \subset \mathbb{R}^n$ is called *closed* if and only if $K^c = \mathbb{R}^n \setminus K$ is open.
(3) If $X \subset \mathbb{R}^n$, then a set $U \subset X \subset \mathbb{R}^n$ is called *open relative to* X or *relatively open in* X if and only if $U = X \cap W$ for some set $W \subset \mathbb{R}^n$ which is open in $\mathbb{R}^n$. Equivalently, for all $a \in U$ there is a ball $B_r(a)$ such that

$$B_r(a) \cap X \subset U.$$

(4) If $X \subset \mathbb{R}^n$, then a set $K \subset X \subset \mathbb{R}^n$ is called *closed relative to* X or *relatively open in* X if and only if $K = X \cap V$ for some set $V \subset \mathbb{R}^n$ which is closed in $\mathbb{R}^n$.

1.2. Descriptions of functions.

Here are some important definitions that describe different types of functions.

As a convenient notation, we shall write

$$f : A \to B$$

to mean that f is a function which takes inputs $a \in A$ and give outputs $f(a) \in B$.

Definition 1.2. Let $f : A \to B$.

(1) f is called a *bijection* between A and B if and only if the following condition holds.
For all $b \in B$, there is a unique $a \in A$ such that $f(a) = b$.
(2) f is called *continuous* on A if and only if for all sets U open in B, $f^{-1}(U)$ is open in A.
(3) f is called a *homeomorphism* between A and B if and only if it has the following properies.
   (a) f is a bijection between A and B.
   (b) Both $f : A \to B$ and $f^{-1} : B \to A$ are continuous functions.
(4) f is called a *topological embedding* if $f : A \to f(A)$ is a homeomorphism.

Definition 1.3. A set $X \subset \mathbb{R}^n$ is called *connected* if for all pairs of open sets $U, W \subset \mathbb{R}^n$ the following condition holds.

If $X \cap U \neq \emptyset$ and $X \cap W \neq \emptyset$ and $X \subset U \cup W$ then $U \cap W \neq \emptyset$.

A set $X \subset \mathbb{R}^n$ for which there *does* exist two open sets U, W such that $X \subset U \cup W$, $X \cap U \neq \emptyset$, $X \cap W \neq \emptyset$, and $U \cap W = \emptyset$ is called *disconnected.*

Theorem 1.4. *If $U \subset \mathbb{R}^n$ is connected and $f : U \to \mathbb{R}^m$ is continuous then $f(U)$ is connected.*

Definition 1.5. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$.

- We say
$$\dim_{\mathcal{F}}(X) \leq \dim_{\mathcal{F}}(Y)$$
if there exists a *topological embedding* $f : X \to Y$.
- We say
$$\dim_{\mathcal{F}}(X) = \dim_{\mathcal{F}}(Y)$$
if $\dim_{\mathcal{F}}(X) \leq \dim_{\mathcal{F}}(Y)$ and $\dim_{\mathcal{F}}(Y) \leq \dim_{\mathcal{F}}(X)$.
- We say
$$\dim_{\mathcal{F}}(X) < \dim_{\mathcal{F}}(Y)$$
if $\dim_{\mathcal{F}}(X) \leq \dim_{\mathcal{F}}(Y)$ and $\dim_{\mathcal{F}}(Y) \not\leq \dim_{\mathcal{F}}(X)$.

2. Problem Session #2

2.1. **Computational problems.** The Big Question here is: now that we have a new notion of dimension ($\dim_{\mathcal{F}}$), what is the dimension of various sets?

(1) Draw the following sets
- Let $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
- Let $B = A \cup \{(x, y) \in \mathbb{R}^2 : (x - 3)^2 + y^2 = 1\}$.
- Let $C = A \cup \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 = 1\}$.
- Let $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2 + y^2} \leq 2\}$.
- Let $E = D \cup \{(x, y) \in \mathbb{R}^2 : 1 \leq \sqrt{(x - 6)^2 + y^2} \leq 2\}$.

(2) Now that you have a picture of A, B, C, D, E, consider their Frechet Dimension-type.

Sean's note: For this section, they are NOT required to "write a proof." It is enough to draw the right pictures and be able to explain where the problem comes from.

(a) Show that $\dim_{\mathcal{F}}(A) < \dim_{\mathcal{F}}(B)$.

Sean's note: The embedding $A \hookrightarrow B$ is trivial. The BIG IDEA that prevents the embedding $B \hookrightarrow A$ is that if $A \to A$ is not onto, then it is not a bijection. This argument will have been done in class. But, for completeness, here it is.

Suppose that there is a topological embedding BA. Then, by restriciton, there is a topological embedding of $f : A \to A$ which is NOT surjective. Thus $f(A) \subset A$ is a closed sub-arc. Let x be an endpoint of $f(A)$. By the definition of topological embedding (local homeomorphism) then, for every open ball $B_\rho(f^{-1}(x))$,
$$f : B_\rho(f^{-1}(x)) \to f(B_\rho(f^{-1}(x)))$$
is a homeomorphism. However, for sufficiently small $0 < \rho$ this is impossible since there is no homeomorphism between $(0, 1)$ and $[0, 1)$. (Draw the picture.)

(b) Show that $\dim_{\mathcal{F}}(A) < \dim_{\mathcal{F}}(C)$.

Sean's note: Again, the embedding $A \hookrightarrow C$ is trivial. The point that prevents $C \hookrightarrow A$ is the point $(1,0)$. No continuous map of a neighborhood of $C \cap \{(1,0)\}$ into A can be injective. Again, this uses the same argument as above.

(c) Show that $\dim_{\mathcal{F}}(B)$ cannot be compared to $\dim_{\mathcal{F}}(C)$.

Sean's note: Both the problems in (1) and (2) occur in this problem. $B \not\hookrightarrow C$ because of of the problems from (1). $C \not\hookrightarrow B$ because of the problems from (2).

(d) Show that $\dim_{\mathcal{F}}(D) = \dim_{\mathcal{F}}(E)$. What is the difference between this example and A and B?

Sean's note: The embeddings come from scaling down small enough. Drawing the picture is good enough.

The difference, of course, is that D, E are "higher dimensional."

(3) How does this compare with your intuition of how "dimension" should work? How does this compare with your intuition about what dimension the sets A, B, C, D, E should be?

2.2. **Exploration Problems.** The Big Question in this section is:

> Can we modify our definition of $\dim_{\mathcal{F}}$ to make $\dim_{\mathcal{F}}$ of a circle the same as $\dim_{\mathcal{F}}$ of $\mathbb{R}^1$?

It may help to consider the following questions.

- When you show that $\dim_{\mathcal{F}}(\mathbb{R}^1) \leq \dim_{\mathcal{F}}(\text{circle})$, you show that there is a portion of the circle which is homeomorphic to $\mathbb{R}^1$. What do these pieces look like? Can you find them near every point in the circle?
- The problem in comparing $\dim_{\mathcal{F}}(\mathbb{R}^1)$ and $\dim_{\mathcal{F}}(\text{circle})$ was in mapping *the whole* circle to $\mathbb{R}^n$. What if we gave that up? What might our new definition look like?

If you have a way to modify $\dim_{\mathcal{F}}$ and produce a new definition of "dimension," how does your new definition work on some of the examples A, B, C, D, E in the "Computational Questions," above?

Sean's note: The BIG IDEA I want the students to get is "local parametrization" in the same way that manifolds are defined. That is, something like:

Definition 2.1. Let $0 \leq m \leq n$. A set $X \subset \mathbb{R}^n$ is said to be "m-dimensional" if for every $p \in X$ there exists a neighborhood $p \in U$ such that

$$X \cap U$$

is homeomorphic to $B_1(0) \subset \mathbb{R}^m$.

This works for A, B, D, and E, above.

But, it does NOT work for C.

If participants get this far, **then** please ask them to try to modify their definition again to account for C.

One option (following the notion of rectifiable sets) is to relax parametrization to containment like in the theory of rectifiable sets.

Another option is to relax local parametrization at every point to local parametrization at most points. In fact, no matter what one does, there will always be problems.

- For example, if we let $\partial B_1^p(0)$ be the sphere in the $\ell^p(2) = (\mathbb{R}^2, |\cdot|_p)$, consider

$$E := \{x \in \mathbb{R}^2 : x \in \partial B_1^p(0) \text{ for } p \in \mathbb{N} \cup \{\infty\}\}.$$

No point in $\partial B_1^{\infty}(0)$ is locally homeomorphic to $\mathbb{R}^1$ or $\mathbb{R}^2$.

- Let $E = \{x \in \mathbb{R}^3 : x = (q, 0, 0) \text{ s.t. } q \in \mathbb{Q} \cap [0, 1]\}$.
  No point of E is homoemorphic to $\{0\}$ or $\mathbb{R}^1$.

The big idea in the background is that parametrization has a limitation. For any countable list of model spaces, there is always a set which cannot be locally parametrized any of the listed model spaces.

Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan
Email address: smccurdy@ntnu.edu.tw