WHAT IS DIMENSION? DAY 3

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1. Day 3 Notes

Definition 1.1. Let $U \subset \mathbb{R}^n$. We define the *boundary* of U to be

$$\partial U := \{x \in \mathbb{R}^n : \text{for all } 0 < r, B_r(x) \cap U \neq \emptyset \\ \text{and } B_r(x) \cap (\mathbb{R}^n \setminus U) \neq \emptyset\}.$$

If $U \subset X \subset \mathbb{R}^n$, we define the *boundary of U relative to X* to be

$$\partial^X U = \partial U := \{x \in X : \text{for all } 0 < r, B_r(x) \cap U \neq \emptyset \\ \text{and } B_r(x) \cap (X \setminus U) \neq \emptyset\}$$

Definition 1.2. Let $X \subset \mathbb{R}^n$ and $0 \leq m \leq n$ be an integer.

(1) We define $\dim_T(\emptyset) = -1$.
(2) For $p \in X$, we say
$$\dim_T(X, p) \leq m$$
if anf only if for all open sets $U \subset \mathbb{R}^n$ such that $x \in U$ there exists an open set $x \in V \subset U$ such that
$$\dim_T(X \cap \partial V) \leq m - 1.$$
(3) For $p \in X$ we say that
$$\dim_T(X, p) = m$$
if and only if $\dim_T(X, p) \leq m$ but $\dim_T(X, p) \not\leq m - 1$.
(4) We say that $\dim_T(X) \leq m$ if $\dim_T(X, p) \leq m$ for all $p \in X$.
(5) We say that
$$\dim_T(X) = m$$
if $\dim_T(X) \leq m$ but $\dim_T(X) \not\leq m - 1$.

2. Problem Session #3

2.1. **Computational Questions.** The Big Question in this section is whether or not $\dim_T$ solved some of the problems that $\dim_{\mathcal{F}}$ had.

Consider the following sets

- $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
- $B = A \cup \{(x, y) \in \mathbb{R}^2 : (x - 3)^2 + y^2 = 1\}$.
- $C = A \cup \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 = 1\}$.

(1) What is $\dim_T(A)$?

(2) What is $\dim_T(B)$?
(3) What is $\dim_T(C)$?

Sean's note: Problems (1), (2), (3) all follow the arguement presented in class. Let $X \in \{A, B, C\}$. Basically, because X is locally connected, for any $p \in X$ and for any neighborhood $p \in V \subset U$, $\partial V \cap X \neq \emptyset$. This forces $\dim_T(X) = 1$.

(4) Let $E \subset \mathbb{R}^1$ be the set

$$E := \{x : x = \frac{k}{2^n}; k, n \in \mathbb{N}\}.$$

What is $\dim_T(E)$? What is $\dim_T(\mathbb{R}^1 \setminus E)$? Since $\mathbb{R}^1 = E \cup (\mathbb{R}^1 \setminus E)$, isn't this a bit strange?

Sean's note: $\dim_T(E) = 0$. E is totally disconnected, so we can always find V with $\partial V \cap E = \emptyset$. Same with $\mathbb{R}^1 \setminus E$.

In general, $\mathbb{R}^n = \cup_{i=1}^{n+1} A_i$ for disjoint sets A_i with $\dim_T(A_i) = 0$. But students do NOT need to discover this themselves.

(5) Can you use (4.) to come up with two sets, F, G such that $\dim_T(F) = 1 = \dim_T(G)$ and $\mathbb{R}^2 = F \cup G$?

Sean's note: Take $F = E \times \mathbb{R}^1$ and $G = (\mathbb{R}^1 \setminus E) \times \mathbb{R}^1$.

The proof that these actually have $\dim_T(F) = \dim_T(G) = 1$ comes from taking axis-parallel rectangles.

2.2. **Exploration Questions.** We would like to show that $\dim_T(\mathbb{R}^n) > n - 1$. To do so, we need to explore what the difference between $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ is. So, the Big Question is:

What properties does $\mathbb{R}^n$ have that $\mathbb{R}^{n-1}$ does not?

To help your exploration, you may want to consider the following sub-questions.

(1) What did $\dim_V$ say about the difference between $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$?
(2) How might that relate to boundaries and the intersection of boundaries?

Sean's note: The BIG IDEA is:

In $\mathbb{R}^n$ we can find n linearly independent vectors and in $\mathbb{R}^{n-1}$ we cannot.

In $\mathbb{R}^n$ the mutual intersection of n hyperplanes is (in general) non-empty. But, in $\mathbb{R}^{n-1}$ the intersection of n hyperplanes is (in general) empty.

Using the fact that vectors define hyperplanes, we can relate information about vectors to information about the intersection of hyperplanes. **These hyperplanes are the boundary of half-spaces.**

If students come to this conclusion, then remind them of the intuition that

The intersection of m $n-1$-planes in $\mathbb{R}^n$ has dimension $n - m$.

Ask them how this might relate to the inductive definition of $\dim_T$.

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