Elimination of Cusps in a Free Boundary Problem for Water Waves

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The Euler Equations, Velocity Form

In 2-dimensions, $\vec{v}(x, y, t) = (v_1(x, y, t), v_2(x, y, t))$

$$\begin{aligned} \partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p &= 0 \\ \operatorname{div}(\vec{v}) &= 0 \\ (v_1)_y - (v_2)_x &= 0 \\ \vec{v} \cdot \vec{\eta}_D &= \text{normal speed of } \partial D(t). \\ p &\text{ is locally constant on } \partial D(t). \end{aligned}$$

The Bernoulli Principle

Bernoulli Principle

$$p + \frac{1}{2}|\nabla u|^2 + gy \quad \text{is constant in } D$$

Normalizing, we obtain the following Bernoulli-type free-boundary problem.

$$\begin{cases} \Delta u & = 0 \quad \text{in } \{u > 0\} = D \subset R \\ u & = 0 \quad \text{on } \partial D \cap R \\ |\nabla u|^2 & = -y \quad \text{on } \partial D \cap R. \end{cases} \tag{1}$$

A very abridged History

1847 Stokes conjectures the existence of an extreme wave with peak angle $2\pi/3$.

1880 Stokes gives a formal argument justifying his conjecture [Sto09].

1961 Krasovskiĭ [Kra61] and Keady and Norbury [KN78] prove the existence of smooth waves of large amplitude.

1978 Toland [Tol78] and McLeod [McL97] proved the existence of extreme periodic waves, of infinite and finite depth.

1982 Plotnikov [Plo82] and Amick, Fraenkel, and Toland [AFT82] independently prove the Stokes conjecture under the assumptions:

- The stagnation point is isolated.
- ∂D is symmetric.
- ∂D is monotone.
- ∂D is the graph of a function.
- The stream function u is increasing in y.

A very abridged History

2011 Varvaruca and Weiss introduced the “geometric approach” to the study of the Stokes Wave, and generalized the free-boundary problem to $n \geq 2$.

- Monotonicity formula: the Weiss $\frac{3}{2}$-density.

$$W_{\frac{3}{2}}(x_0, r, u) = \frac{1}{r^{n-2+2(\frac{3}{2})}} \int_{B_r(x_0)} |\nabla u|^2 + x_n^{2(\frac{3}{2}-1)} \chi_{\{u>0\}} dx - \frac{\frac{3}{2}}{r^{n-1+2(\frac{3}{2})}} \int_{\partial B_r(x_0)} u^2 d\sigma.$$

- Blow-ups.
- Geometric Measure Theory.

[VW11]

Theorem ([VW11], Proposition 5.5 and Proposition 5.8)

Let $n \geq 2$, and let u is a weak solution to (1) satisfying $|\nabla u|^2 \leq C|x_n|$ locally in $D \subset \mathbb{R}^n$. Define,

$$S := \{x \in \{x_n = 0\} \cap \partial D : 0 < \lim_{r \to 0} \frac{\mathcal{H}^n(B_r(x) \cap \{u > 0\})}{\omega_n r^n} < \frac{1}{2}\}$$

$$S_{\frac{1}{2}} := \{x \in \{x_n = 0\} \cap \partial D : \lim_{r \to 0} \frac{\mathcal{H}^n(B_r(x) \cap \{u > 0\})}{\omega_n r^n} = \frac{1}{2}\}.$$

Then, $S \cup S_{\frac{1}{2}}$ satisfies $\dim_{\mathcal{H}}(S \cap S_{\frac{1}{2}}) \leq n - 2$. Furthermore, if $n = 2$, then S is locally isolated and every point in S satisfies the Stokes conjecture.

What about Cusps?

Theorem ([VW11] Lemma 4.4)

If $n = 2$ and u is a weak solution to (1) satisfying $|\nabla u|^2 \leq |x_n|$ locally in $D \subset \mathbb{R}^n$ then $S_0 = \emptyset$, where

$$S_0 := \{x \in \{x_n = 0\} \cap \partial D : \lim_{r \to 0} \frac{\mathcal{H}^n(B_r(x) \cap \{u > 0\})}{\omega_n r^n} = 0\}.$$

The Question of this Talk

1. Do cusps happen?
2. How do we get information about solutions near cusps?

Local Minimizers: an easier case?

Bernoulli-type Free-boundary Problems

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap \Omega \subset \mathbb{R}^n \\ |\nabla u(x)| = Q(x) & x \in \partial\{u > 0\} \cap \Omega \\ u = u_0 & \text{on } \partial\Omega, \end{cases}$$

arise as the Euler-Lagrange Equations for functionals of the form

$$\mathcal{J}_\Omega(u, Q) := \int_\Omega |\nabla u(x)|^2 + Q^2(x)\chi_{\{u>0\}}(x) d\mathcal{H}^n(x).$$

The Comparison Principle

Theorem ([AC81] Lemma 3.2)

Let u be a local minimizer of $\mathcal{J}_Q(\cdot, B_2(0))$. There is a constant, $C_{max}(n) > 0$ such that for every ball, $B_r(x) \subset B_2(0)$, if

$$\frac{1}{r^{n-1}} \int_{\partial B_r(x)} u d\sigma > C_{max} r \cdot \max_{y \in B_r(x)} Q(y),$$

then $u > 0$ in $B_r(x)$.

Theorem ([AC81] Lemma 3.4)

Let u be an ϵ_0-local minimizer of $\mathcal{J}_Q(\cdot, \Omega)$. Let $s \in (0, 1)$ be fixed. Then, for all $0 < r \leq \frac{1}{2} r_0$ and all $B_{2r}(x_0) \subset \Omega$, there is a constant $C_{\min} = C(n, s)$ such that if

$$\frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u d\sigma \leq r C_{\min} \min_{y \in B_{sr}(x_0)} \{Q(y)\},$$

then $u = 0$ on $B_{sr}(x_0)$.

Absolute Minimizers: [AL12], [GL18], [GL19]

[AL12] Investigated absolute minimizers of $\mathcal{J}_\Omega(u, Q)$ (under appropriate boundary data) and obtain a "local" analog of [KN78]. But, they are unable to obtain a Stokes wave.

[GL18] Investigated absolute minimizers of $\mathcal{J}_\Omega(u, Q)$ (under appropriate boundary data) and obtain non-flat profiles in dimensions $n \geq 2$ and arbitrary exponent.

Local Minimizers in a more general setting

Theorem ([McC20])

Let $0 \leq k \leq n-1$*. Let* $\Gamma \subset \mathbb{R}^n$ *be a k-dimensional* $(1, M)$*-*$C^{1,\alpha}$ *submanifold such that* $0 \in \Gamma$*, and let* $0 < \gamma$*. Let* $Q(x) = dist(x, \Gamma)^{\gamma}$*, and let u by a local minimizer of* $\mathcal{J}_{B_2(0)}(\cdot, Q)$*.*

Then, $\{u > 0\} \cap B_1(0)$ *is a set of finite perimeter. In particular, the "cusp" set* S_0 *has* $\mathcal{H}^{n-1}$*-measure zero.*

The Big Idea

Instead of looking for an analytic tool to apply at the tip of a cusp, let's examine the geometric behaviour of solutions near the tip of a cusp.

Growth of Harmonic Functions

Local geometry of cusps

Definition

Property P Let $u : B_2^n(0,0) \to \mathbb{R}^+$ be continuous. For an integer $N \in \mathbb{N}$, a ball $B_r^n(x,0) \subset B_2^n(0,0)$, is said to satisfy Property P with constant N if

$$B_r^k(x) \times B_{\frac{2r}{N}}^{n-k}(0) \cap \{u > 0\}$$

has a component $\mathcal{O}$ which satisfies the following conditions.

1. (Upper Height Bound)

$$\sup_{0 \le \rho \le r} \sup\{|y| : (x', y) \in \mathcal{O}, |x' - x| = \rho\} \le \frac{r}{N}.$$

2. (Lower Height Bound)

$$\inf_{0 \le \rho \le r} \sup\{|y| : (x', y) \in \mathcal{O}, |x' - x| = \rho\} \ge \frac{r}{N4 \cdot 2^{1+\gamma}}.$$

The contradiction

Let $0 < \epsilon_0$, and let n, k be integers such that $n \geq 2$ and $0 \leq k \leq n-1$. Let $\Gamma = \{(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\}$, and $0 < \gamma$. Let $Q(x, y) = |y|^\gamma$. If u is an ϵ_0-local minimizer of $J_Q(\cdot, B^n_N(0, 0))$ and $B^n_N(0, 0)$ has Property P with constant N, then

1. By Free-boundary estimates, for all $1 \leq r \leq N$
$$\int_{\partial B_r(x)} u^2 \approx_{n,\gamma} 1.$$
2. By Harmonic estimates,
$$\int_{\partial B_R(x)} u^2 d\sigma \geq \left(\frac{R}{r}\right)^{n-1} \int_{\partial B_r(x)} u^2 d\sigma.$$

Local Minimizers and Cusps: Results

Theorem (McCurdy-Naples 2021)

Let $0 < \epsilon_0$, and let n, k be integers such that $n \geq 2$ and $0 \leq k \leq n-1$. Let $\Gamma = \{(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\}$, and $0 < \gamma$. Let $Q(x, y) = |y|^\gamma$. If u is an ϵ_0-local minimizer of $J_Q(\cdot, B_2^n(0, 0))$, then $S_0 = \emptyset$.

What about Critical Points?

What about Critical Points? Ingredient 1

The Divergence Theorem

If u is a weak solution to (1), this assumes some regularity on $\partial\{u > 0\}$ away from $\{y = 0\}$ which implies

$$\int_{\partial\mathcal{O}\cap[-1,1]^2} \nabla u \cdot \eta d\sigma = -\int_{\mathcal{O}\cap\partial[-1,1]^2} \nabla u \cdot \eta d\sigma$$

What about Critical Points? Ingredient 2

If u is a weak solution to (1), $(x,0)$ is a cusp point, and $(x,0) \in \partial\mathcal{O}$ then for every $0 < C_2 \leq 1$ there is a rescaling $\mathcal{O}_{(x,0),\rho}$ such that

$$\int_{\partial\mathcal{O}_{(x,0),\rho}\cap[-1,1]^2} |x_2|^\gamma d\sigma \geq \int_{\mathcal{O}_{(x,0),\rho}\cap\partial[-1,1]^2} \frac{1}{6C_2}|x_2|^\gamma d\sigma.$$

Critical Points Results

Theorem (McC22)

Let $C < \infty$, and u is a weak solution to (1) satisfying $|\nabla u|^2 \leq Cx_2^+$ locally in $D \subset \mathbb{R}^2$. Then,

- $S_0 = \emptyset$.
- *No point of $\mathcal{S}$ can also be a cusp point.*

The End

Thank you

Bibliography

H. W. Alt and L. A. Caffarelli.
Existence and regularity for a minimum problem with free boundary.
J. Reine Angew. Math., 325:105–144, 1981.

C. J. Amick, L. E. Fraenkel, and J. F. Toland.
On the Stokes conjecture for the wave of extreme form.
Acta Math., 148:193–214, 1982.

D. Arama and G. Leoni.
On a variational approach for water waves.
Comm. Partial Differential Equations, 37(5):833–874, 2012.

Giovanni Gravina and Giovanni Leoni.
On the existence of non-flat profiles for a bernoulli free boundary problem, 2018.

Giovanni Gravina and Giovanni Leoni.
On the behavior of the free boundary for a one-phase bernoulli problem with mixed boundary conditions, 2019.

G. Keady and J. Norbury.
On the existence theory for irrotational water waves.