Here are some things to work on. Don't worry about finishing them. Just get as far as you can in 4 hours. But definitely read and think about the Symbol spaces one. Even if you don't get anywhere, read it and think about it.

I would like to see your proofs and hear your ideas ideas on Thursday. We can discuss any questions you have, too. It is OK to come with lots of ideas and questions, but no finished proofs. I do that all the time with my advisor. But try to get as far as you can in the time you have.

0.1 Cardinality and Paradoxicality

Definition 0.1. (G-Paradoxical) Let G be a group acting on a set X. X is called G-paradoxical (or, paradoxical with respect to G) if for some positive integers, n, m, there are pairwise disjoint subsets

$$A_1, A_2, ..., A_n, B_1, B_2, ..., B_m$$

of E and group elements $g_1, g_2, ..., g_n, h_1, h_2, ..., h_m \in G$ such that $X = \bigcup_{g_i} g_i(A_i)$ and $X = \bigcup_{h_i} h_i(B_i)$.

Let X be a set and $|X|$ denote the cardinality of X.

1. Show that the set of bijections, $\{b | b : X \to X \text{ is a bijection}\}$, is a group. Note that $\{b | b : X \to X \text{ is a bijection}\}$ is a group which "acts upon" X.

2. Let $X = \mathbb{N}$, the natural numbers. Show that
   - $|\mathbb{N}| = 2|\mathbb{N}|$.
   - $\mathbb{N}$ is paradoxical with respect to $\{b | b : X \to X \text{ is a bijection}\}$.

   Feel free to prove one, and then use that one to prove the other.

0.2 Symbol Spaces

Consider the set of one-sided infinite strings,

$$\{0,1\}^{\mathbb{N}} = \{(x_0, x_1, x_2, ...) | x_i \in \{0,1\}\}.$$

1. Show that $|\{0,1\}^{\mathbb{N}}| = |[0,1]|$. That is, construct a bijection from $\{0,1\}^{\mathbb{N}}$ onto $[0,1]$. *HINT: the big idea is that*
   - *every string which begins with* 0 *corresponds to a point in* $[0, \frac{1}{2}]$,
   - *every string which begins with* 1 *corresponds to a point in* $[\frac{1}{2}, 1]$,
   - *every string which begins with* 00 *corresponds to a point in* $[0, \frac{1}{4}]$,
   - *every string which begins with* 01 *corresponds to a point in* $[\frac{1}{4}, \frac{1}{2}]$,

   and so on and so forth. You have to worry about the set of strings which end in repeating 0*s or repeating* 1*s, but this is countable, and can be shifted around Hilbert Hotel style.*

2. Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift map which deletes the first entry. That is,

$$\sigma((x_0, x_1, x_2, ...)) = (x_1, x_2, x_3, ...).$$

Show that $\{0,1\}^{\mathbb{N}}$ is paradoxical in the sense that there are four mutually disjoint subsets, $A_1, A_2, B_1, B_2 \subset \{0,1\}^{\mathbb{N}}$ such that

$$\{0,1\}^{\mathbb{N}} = \sigma(A_1) \cup \sigma(A_2)$$

and

$$\{0,1\}^{\mathbb{N}} = \sigma(B_1) \cup \sigma(B_2).$$