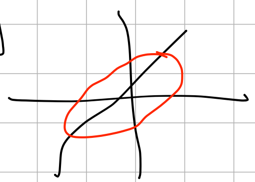
$\boxed{\text{Lecture II:}}$

In Lecture I, we saw a version of dimension which matched our intuition for $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots$ but which <u>only</u> worked for vector spaces (lines, planes, etc.)

We want to develop a notion of dimension that applies to things which are <u>not</u> lines, planes, etc.

$\boxed{\text{Ex}}$



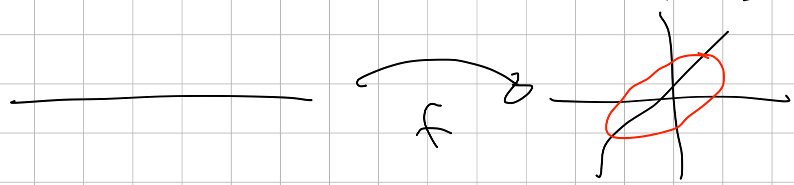
$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1, \ z=0\}$$

- What dimension do we think C is? 3? 2? 1?
- Is kinda like $\mathbb{R}$.

<u>Q:</u> In what sense is C like $\mathbb{R}$?

One sense is that there is a function

$$f(t) = (\cos(t), \sin(t), 0)$$

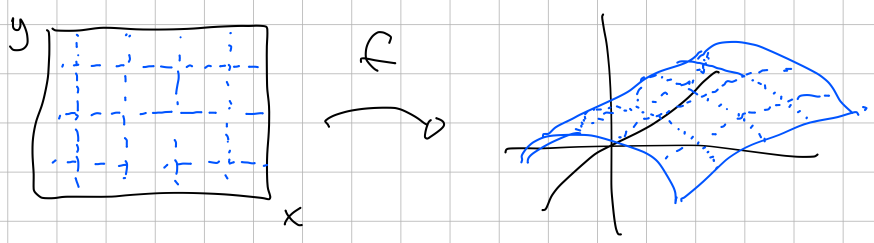


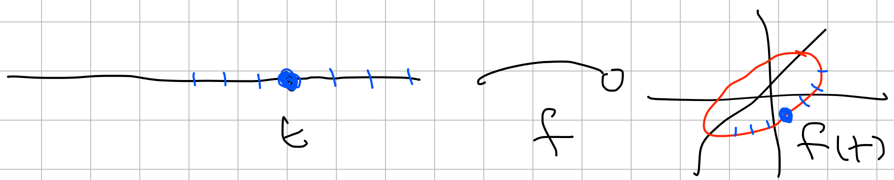
We can think of f as <u>parametrizing</u> C.

<u>Def:</u> <u>Parametrization</u> involves

$$\boxed{\text{Model Space/ Parameter Space}} \xrightarrow{\text{function}} \boxed{\text{Thing being parametrized.}}$$

If the model space is $\mathbb{R}^n$ we can think of the coordinates in the model space as giving "coordinates" to the thing being parametrized.





Coordinates in the model space are called <u>parameters</u>.

One way to <u>extend</u> our definition of dimension is by <u>parametrization</u>.

Idea

We might say that if we can parametrize a set X by a model space $\mathbb{R}^n$ then dimension of X should be n.

We know dimension of model spaces $(\mathbb{R}, \mathbb{R}^2, \dots)$

We hope to extend this by means of the <u>parametrizing function</u>.

Another way to think of this is that for "nice" f, $\mathbb{R}^n$ and X are <u>equivalent</u> from the perspective of dimension.

<u>or</u>

"Dimension" is <u>preserved</u> by the parametrizing function.

Therefore, in seeking to <u>extend</u> our notion of dimension by <u>parametrization</u>, we need to find a suitable class of parametrizing functions that preserve enough properties to justify preserving "dimension".

$\boxed{\text{Attempt \#1}}$ what if we preserve the "number of points"?

<u>Def:</u> a function $f: X \to Y$ is called a <u>bijection</u> <u>iff</u> for all $y \in Y$ there is a unique $x \in X$ such that $f(x) = y$.

$$X \xrightarrow{f} Y$$

Maybe if there exists a bijection $f: X \to \mathbb{R}^n$ then dimension $X = n$.

$\boxed{\text{Problem}}$ In the 19th Century, Cantor showed there exists a bijection $f: [0,1] \to \mathbb{R}^n$. This would imply $1 = \dim([0,1]) = \dim(\mathbb{R}^n) = n$.

→ we cannot use bijections.

we need to use a smaller class of parametrizing functions.

one that preserves more properties.

$\boxed{\text{Attempt \#2}}$ What if we ask for continuity? Equivalence → homeomorphism

Def: a function $f: X \to Y$ is called a homeomorphism if f is a bijection and both, f, f^{-1} are continuous.

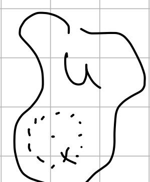
easy: draw graph without lifting pencil.

Q: What if we use homeomorphisms as our parametrizing functions?

In order to see if this class will work, we need to study continuous functions. $f: X \to Y$

A Brief Introduction to Topology.

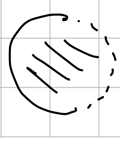
Def: a set $U \subseteq \mathbb{R}^n$ is "open" if for all $x \in U$, $\exists B_{r_x}(x) \subset U$.



a set $K \subseteq \mathbb{R}^n$ is "closed" if $K^c = \mathbb{R}^n \setminus K$ is open.

$\boxed{\text{Ex}}$ $B_1(0) = \{ x \in \mathbb{R}^n \mid |x| < 1 \}$ is open

$\overline{B_1(0)} = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \}$ is closed.

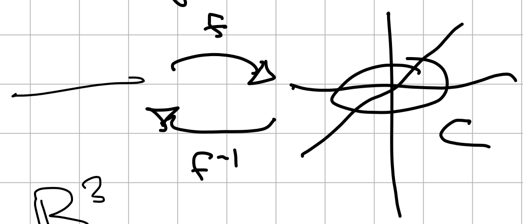
 neither open nor closed.

Def: a function $f: \mathbb{R}^n \to \mathbb{R}^m$

is "continuous" if for all

U open in $\mathbb{R}^m$, $f^{-1}(U)$ is open in $\mathbb{R}^n$.

$\boxed{\text{Ex}}$ Consider our example



$f: \mathbb{R} \to \mathbb{R}^3$

- we know what open sets in $\mathbb{R}^3$ are.
- we know what open sets in $\mathbb{R}$ are.

But for $f^{-1}: C \to \mathbb{R}$

- we know what open sets in $\mathbb{R}$ are
- what are open sets in C?

Def: Let $X \subseteq \mathbb{R}^n$.

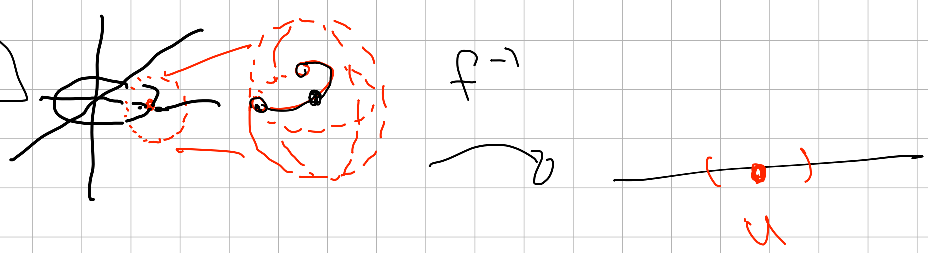
a set $U \subseteq X$ is open relative to X

if $U = X \cap A$ for A open in $\mathbb{R}^n$

a set $K \subseteq X$ is closed relative to X

if $K = X \cap B$ for B closed in $\mathbb{R}^n$.

$\boxed{\text{Ex}}$



Now we can talk about continuous functions between subsets in $\mathbb{R}^n, \mathbb{R}^m$.

Def: for $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$,

$f: X \to Y$ is continuous if for all

U relatively open in Y, $f^{-1}(U)$ is relatively open in X.

All of this was just so that we have the infrastructure to talk about homeomorphisms between subsets $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$.

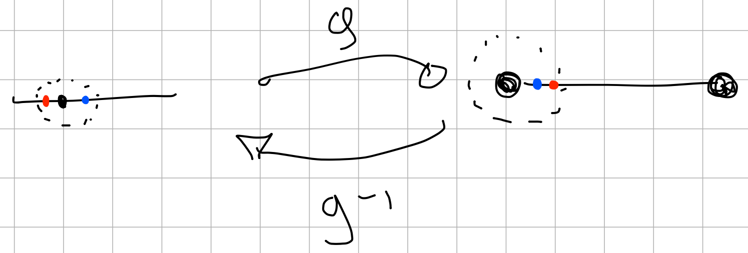
Now that we know what that means, we return to the question:

"are homeomorphisms a good class of parametrizing functions?"

Let's consider an easy example.

$\boxed{\text{Ex}}$ Is there a bicontinuous bijection $g: \mathbb{R} \to [0,1]$?

Q: What happens to $g^{-1}(0)$?

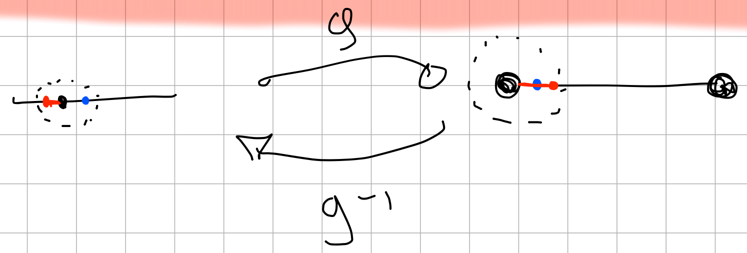


Big Theory Fact: If $U \subseteq \mathbb{R}^n$ is connected

(for all open sets A, B s.t. $U \cap A \neq \emptyset$
$U \cap B \neq \emptyset$
$U \subset A \cup B$, $A \cap B \neq \emptyset$)

and $f: U \to \mathbb{R}^n$ is continuous,

Then $f(U)$ is connected.



This contradicts g a bijection!

$[0,1]$ is not homeomorphic to $\mathbb{R}$.

If we want to define $\dim([0,1]) = \dim(\mathbb{R})$ by parametrization, we cannot use homeomorphisms.

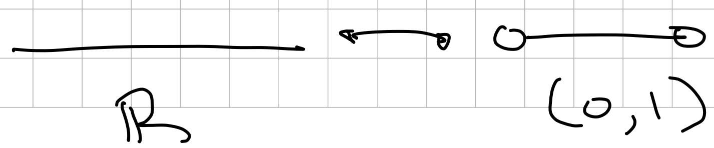
We need a less restrictive class.

$\to$ functions which preserve fewer properties.

If we look back at our example:

we see that the endpoints are the problem.

Indeed, if we remove the endpoints,



then $\exists\, f : \mathbb{R} \to (0,1)$ a homeomorphism!

$\boxed{\text{Ex}}$ $f : \mathbb{R} \to (0,1)$ $\quad f(x) = \frac{1}{\pi}\tan^{-1}(x) + \frac{1}{2}$

is a homeomorphism.

Big Idea: (Localization) What if instead of

<u>all</u> of X being "equivalent" to <u>all</u> of Y

we just ask that

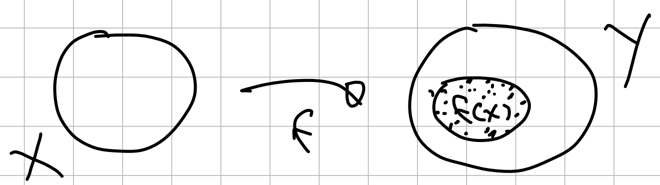
1. <u>part</u> of Y is equivalent to <u>all</u> of X
2. <u>part</u> of X is equivalent to <u>all</u> of Y. ?

(Give up global bijection)

Attempt #3: (Partial localization)

Def: $f : X \to Y$ is a <u>topological embedding</u>

if $f : X \to f(X)$ is a homeomorphism.



$\boxed{\text{Ex}}$ $f : \mathbb{R} \to [0,1]$ $\qquad g : [0,1] \to \mathbb{R}$

$f(x) = \frac{1}{\pi}\tan^{-1}(x) + \frac{1}{2}$ $\qquad g(x) = x$

- $\mathbb{R}$ and $[0,1]$ can both be topologically embedded into each other.

__Def__: Frechet Dimension-type.

Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$

$\boxed{1.}$ we say $\dim_F(X) \leq \dim_F(Y)$

if $\exists f: X \to Y$ a top. embedding.

$\boxed{2.}$ we say $\dim_F(X) = \dim_F(Y)$

if $\dim_F(X) \leq \dim_F(Y)$

and $\dim_F(Y) \leq \dim_F(X)$.

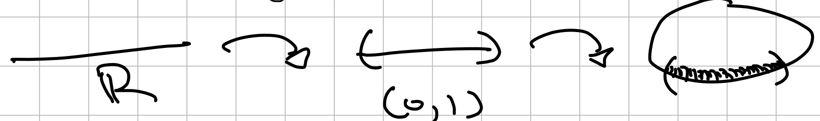
__Good News__ $\dim_F((0,1)) = \dim_F(\mathbb{R}) = \dim_F(\;) = \dim_F(\text{graph continuous } f)$

$\to$ better than homeomorphisms!!

__Strange__: $\dim_F(\mathbb{R})$ is not 1, but dimensions are defined relative to other spaces...

To see why, consider our example.

- Can we topologically embed $\mathbb{R}$ in C?

yes! 

- Can we topologically embed $C \to \mathbb{R}$?

Suppose we could.

$g : \mathbb{C} \to \mathbb{R}$. a homeomorphism onto $g(\mathbb{C})$.

$g : \mathbb{C} \to g(\mathbb{C})$ continuous, bijection

$g^{-1} : g(\mathbb{C}) \to \mathbb{C}$ continuous.

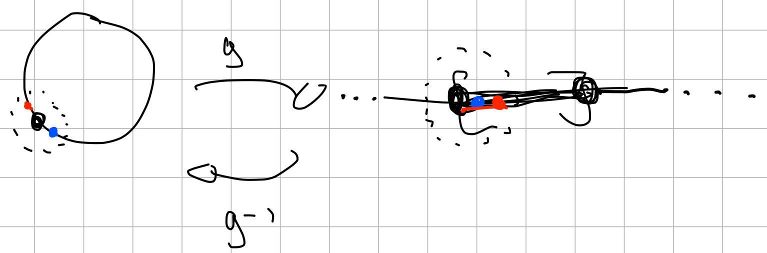
- Since $\mathbb{C}$ is closed, it turns out

  $g(\mathbb{C})$ must be closed.

- By our big theory fact,

  g cannot separate / disconnect $\mathbb{C}$,

  so $g(\mathbb{C})$ is a closed interval $[a, b]$



We obtain the same contradiction as before.

so $\dim_F(\mathbb{R}) < \dim_F(\mathbb{C})$

<u>and</u> It turns out,

$\dim_F(\mathbb{C}) < \dim_F(\mathbb{R}^2)$.

(but this is much harder).

If we want to think of $\dim(\mathbb{R}^1) = \dim(\mathbb{C})$

we cannot use <u>topological embedding</u>.

We need a more flexible notion.

$\boxed{\text{Take Home Question}}$

Can you come up with a better sense of "local equivalence" using parametrizing functions and use it to build a definition of "dimension" such that $\dim(\mathbb{C}) = \dim(\mathbb{R})$?