WHAT IS DIMENSION? DAY 5

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1. Day 5 Notes

Definition 1.1. For any bounded set $X \subset \mathbb{R}^n$ we define

$$\dim_{\mathcal{M}}(X) = \lim_{r \to 0} \frac{\log(N_x(r))}{\log(1/r)}$$

Definition 1.2. For a function $f : \mathbb{R} \to \mathbb{R}$ we say that

$$\lim_{r \to 0} f(r) = a$$

if and only if for all $0 < \epsilon$ there exists a $0 < \delta$ such that if $|0 - \rho| < \delta$ then $|f(\rho) - a| \leq \epsilon$.

2. Problem Set # 5

Sean's note: These problems are significantly more challenging than previous Problem Sessions. I DO NOT expect participants to be able to get very far without help.

I leave it up to you how to best conduct the session. Feel free to omit any problem you like.

2.1. Computation Problems.

The Big Question in this section is:

Now that we have a new definition of "dimension," how does it behave? What is $\dim_{\mathcal{M}}$ of various sets?

(1) Fix $0 < \alpha$ and let E_α be the set

$$E_\alpha = \{\frac{1}{n^\alpha} : n \in \mathbb{N}\}.$$

Determine $\dim_{\mathcal{M}}(E_\alpha)$.

Sean's note: $\dim_{\mathcal{M}}(E_\alpha) = \frac{1}{1+\alpha}$

We must calculate the n beyond which the balls $B_r(n^{-\alpha})$ and $B_r((n+1)^{-\alpha})$ are connected. Then, we need to count the number of B_r required to cover the remaining points.

Step 1: Find the largest $n \in \mathbb{N}$ such that

$$\frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} \geq r.$$

We calculate

$$\frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} = \frac{(n+1)^\alpha - n^\alpha}{(n+1)^\alpha n^\alpha}.$$

Now, for very small r we see that we may take n very large. And, for large n

$$\begin{aligned}(n+1)^\alpha - n^\alpha &\cong (n^\alpha + \alpha(n)^{\alpha-1} + \alpha(\alpha-1)n^{\alpha-2}) - n^\alpha \\ &\cong C\alpha(\frac{1}{n})^{1-\alpha}\end{aligned}$$

for large n.

Thus, we need to estimate

$$\frac{C\alpha(\frac{1}{n})^{1-\alpha}}{(n+1)^\alpha n^\alpha} = \frac{C\alpha}{(n+1)^\alpha n} \geq r.$$

Equivalently, $\frac{C\alpha}{r} \geq n(n+1)^\alpha$.

Now, we estimate that for large n, $n(n+1)^\alpha \cong n^{1+\alpha}$, whence we estimate

$$C\alpha r \gtrsim n^{1+\alpha}$$

implies that $n \lesssim (\frac{1}{r})^{\frac{1}{1+\alpha}}$. Thus, the largest such n is roughly $n \cong (\frac{1}{r})^{\frac{1}{1+\alpha}}$.

Step 2: We now need to estimate the number of B_r balls required to cover $\{\frac{1}{n^\alpha} : n \geq (\frac{1}{r})^{\frac{1}{1+\alpha}}\}$.

Note that for $n = (\frac{1}{r})^{\frac{1}{1+\alpha}}$

$$\frac{1}{n^\alpha} \cong r^{\frac{\alpha}{\alpha+1}}.$$

Therefore, the diameter of the set $\{\frac{1}{n^\alpha} : n \geq (\frac{1}{r})^{\frac{1}{1+\alpha}}\}$ is $r^{\frac{\alpha}{\alpha+1}}$. Therefore, to cover this set with balls B_r requires at least

$$\frac{r^{\frac{\alpha}{\alpha+1}}}{r} = r^{-\frac{1}{\alpha+1}} = (\frac{1}{r})^{\frac{1}{\alpha+1}}.$$

Step 3: Therefore we estimate that for $0 < r$ sufficiently small $N_E(r) \cong (\frac{1}{r})^{\frac{1}{\alpha+1}} + (\frac{1}{r})^{\frac{1}{\alpha+1}}$.

Thus, we calculate

$$\frac{\log(N_E(r))}{\log(1/r)} \cong \frac{\frac{1}{\alpha+1}\log(1/r)}{\log(1/r)} = \frac{1}{\alpha+1}$$

Since all the estimate become more accurate as $r \to \infty$, we obtain the answer.

(2) This problem concerns the famous Cantor middle third set. This set is constructed as follows.

Let $I_0 = [0,1]$. Now, assume that I_i has been defined. Define I_{i+1} the be what remains after removing the middle $1/3$ interval from all the line segments in I_i.

Let $C(1/3)$ be the limit of this process. Determine $\dim_{\mathcal{M}}(C(1/3))$. What if, instead of the middle $1/3$, we remove the middle $1/7$? What is the dimension of the resulting set?

Sean's note: Covering $C(1/3)$ by balls of radii $r_i = \frac{1}{3^i}$ we see that we require 2^i many such balls.

Now, we need to estimate i as a function of r

$$r_i = \frac{1}{3^i} \quad \to \quad i = \frac{\log(1/r_i)}{\log(3)}.$$

Thus, $N_{C(1/3)}(r_i) \cong 2^i = 2^{\frac{\log(1/r_i)}{\log(3)}}$. Therefore, we observe that

$$\begin{aligned}\lim_{r\to 0}\frac{\log(N_{C(1/3)}(r_i))}{\log(1/r_i)} &= \lim_{r\to 0}\frac{\log(2^{\frac{\log(1/r_i)}{\log(3)}})}{\log(1/r_i)} \\ &= \lim_{r\to 0}\frac{\log(1/r_i)}{\log(3)}\frac{\log(2)}{\log(1/r_i)} = \frac{\log(2)}{\log(3)} < 1.\end{aligned}$$

If we follow these calculations for $C(1/7)$ we obtain $N_{C(1/7)}(r) \cong 2^{\frac{\log(1/r)}{\log(7)-\log(3)}}$ and hence

$$\begin{aligned}\lim_{r\to 0}\frac{\log(N_{C(1/7)}(r))}{\log(1/r)} &= \lim_{r\to 0}\frac{\log(2^{\frac{\log(1/r)}{\log(7)-\log(3)}})}{\log(1/r)} \\ &= \lim_{r\to 0}\frac{\log(1/r)}{\log(7)-\log(3)}\frac{\log(2)}{\log(1/r)} = \frac{\log(2)}{\log(7)-\log(3)}.\end{aligned}$$

(3) In class, we saw the following theorem.

Theorem. For every $0 \le s \le n$ there exists a set $X \subset \mathbb{R}^n$ such that $\dim_{\mathcal{M}}(X) = s$.

Use $1), 2)$ to verify the theorem.

Sean's note: There are several ways to tackle this problem. The easiest way is probably the following.

Step 1: let $s = k + \alpha$ for $k \in \mathbb{N}$ and $\alpha \in [0, 1)$.

Step 2: Use (1) or (2) to obtain a set E of $\dim_{\mathcal{M}}(E) = \alpha$.

Step 3: Now form $E \times \mathbb{R}^k$.

(4) Try to construct a set $X \subset \mathbb{R}^n$ where $\dim_{\mathcal{M}}(X)$ does not exist. It may help to think about Computational Problem 2).

Sean's note: This one is probably too complicated for participants to do in any rigor. The BIG IDEA is to vary the size of intervals we remove in the Cantor construction. By removing middle thirds, we obtain a sequence of r such that

$$\frac{\log(N_X(r))}{\log(1/r)} \cong \frac{\log(2)}{\log(3)}.$$

Then, remove middle sevenths for a sequence of scales until

$$\frac{\log(N_X(r))}{\log(1/r)} \cong \frac{\log(2)}{\log(7) - \log(3)}.$$

Repeat this, alternating as needed so that $lim_{r \to 0} \frac{\log(N_X(r))}{\log(1/r)}$ does not exist.

2.2. Exploration Problems.

The Big Question is:

How do we modify the definition of $\dim_{\mathcal{M}}$ to produce a new definition of "dimension" which

- Only depends upon the *local* properties of a set.
- Exists for all subsets $X \subset \mathbb{R}^n$.

If may help to consider the usual model spaces $(\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \mathbb{S}^1)$ to develop intuition. Once you come up with a new notion of dimension, what does it say the dimension of $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is?

Sean's note: This is an open-ended exploration, and I would like people who participate in it to focus on the joy of exploration. But there are several definitions of "dimension" which participants can be guided towards.

Definition 2.1. (Assouad Dimension) For a non-empty set $X \subset \mathbb{R}^n$, we can define

$$\begin{aligned} \dim_A(X) = \inf\{\alpha :& \text{there exists a constant } 0 < C \text{ such that,} \\ & \text{for all } 0 < r < R \text{ and } x \in X \\ & N_{X \cap B_R(x)}(r) \le C(\frac{R}{r})^\alpha\}. \end{aligned}$$

Or, an even more local version:

Definition 2.2. (Assouad-Nigata Dimension) For a non-empty set $X \subset \mathbb{R}^n$, we can define

$$\begin{aligned} \dim_{A,loc}(X) = \inf\{\alpha :& \text{there exists a constant } 0 < C, 0 < \rho \text{ such that,} \\ & \text{for all } 0 < r < R \le< \rho \text{ and } x \in X \\ & N_{X \cap B_R(x)}(r) \le C(\frac{R}{r})^\alpha\}. \end{aligned}$$

These notions of dimension always exists but may disagree for unbounded sets!

$$\dim_A(\mathbb{N}) = 1 \text{ and } \dim_{A,loc}(\mathbb{N}) = 0.$$

They are a kind of local upper Minkowski dimension. There are obvious corresponding local lower Minkowski dimensions.

Definition 2.3. (Lower Dimension or "minimal dimension number") For a non-empty set $X \subset \mathbb{R}^n$, we can define

$$\begin{aligned}\dim_L(X) = \sup\{\alpha :&\text{there exists a constant } 0 < C \text{ such that,}\\ &\text{for all } 0 < r < R \leq \operatorname{diam}(X) \text{ and } x \in X\\ &N_{X\cap B_R(x)}(r) \geq C(\frac{R}{r})^\alpha\}.\end{aligned}$$

These definitions look like they are local. But it depends upon the local behavior of the worst point $x \in X$. For example, if we let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$,

$$\dim_{\mathcal{M}}(X) = 1/2, \qquad \dim_A(X) = \dim_{A,loc}(X) = \dim_L(X) = 1.$$

This comes from the behavior around $\{0\}$.

Alternatively, we can define a "local version of Minkowski dimension" by chopping our set up.

Definition 2.4. (Packing Dimension) For any set $X \subset \mathbb{R}^n$, we define

$$\dim_P(X) = \inf\{\dim_{\mathcal{M}}(E_i) : X = \cup_i^\infty E_i\}$$

where the infimum is taken over all decompositions $X = \cup_i^\infty E_i$.

This definition is local and gives us the desired result that $\dim_P(\{0\} \cup \{1/n : n \in \mathbb{N}\}) = 0$. But, it depends upon $\dim_{\mathcal{M}}$, so it may or may not exist.

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