Summary:

Today is the last class, so I want to do things backwards and begin with a little summary of what we have done so far:

Mathematically, we started with

$\dim_V$ and the intuition "number of directions we can wiggle."

and then we tried to extend it

with local parametrization in $\dim_F$ and manifolds.

Yesterday we saw two different ideas in topological dimension theory: $\dim_T$ and $\dim_{LC}$.

Each time we produced a new definition of "dimension," studied how the world looked from that perspective, found some strengths and some weaknesses, which inspired us to come up with new definitions to overcome those shortcomings.

Today we are going to see the beginnings of metric dimension theory (analysis), but just because today is the last class, the story strengths / weaknesses → new definitions / theories DOES NOT STOP. I will try to indicate some of the directions this story goes when we get there, just to suggest avenues for further development.

But for now, lets begin with the covering number.

Def: for a bounded set $X \subseteq \mathbb{R}^n$, the "covering number" of X is the smallest number of open balls $B_r(x)$ needed to cover X.

Q: why bounded? If $X \subseteq B_R(0)$, then $N_X(r) < \infty$

If X is unbounded → $N_X(r) = \infty$

$\boxed{Ex}$

1. $X = [0,1]$ $\quad N_X(r) = 1 \quad r > \frac{1}{2}$

$\quad \approx \frac{1}{r} \quad r \leq \frac{1}{2}$

2. $X = [0,1]^2$ $\quad N_X(r) = 1 \quad r > \sqrt{2}$

$\quad \approx \frac{1}{r^2} \quad r \leq \sqrt{2}$

The trick = $\frac{n\text{-Volume of } X}{n\text{-Volume of } B_r(x)}$ → $N_{[0,1]^n}(r) \approx \frac{1}{r^n}$ r small

This is our intuition: the rate of growth in the number of balls we need to cover a space ~ dim.

$\boxed{\text{Algebra}}$ If $N_x(r) = \frac{c}{r^n}$ isolate n.

$$\log(N_x(r)) = \log(c) - \log(r^n)$$

$$\log(N_x(r)) = \log(c) - n\log(r)$$

$$\log(N_x(r)) - \log(c) = -n\log(r)$$

$$\log(N_x(r)) - \log(c) = n\log\left(\tfrac{1}{r}\right)$$

$$n = \frac{\log(N_x(r)) - \log(c)}{\log(1/r)}$$

Since we only care what happens as $r \to 0$

Notice that as $r \to 0$

$$\frac{1}{r} \to \infty$$

$$\log\left(\tfrac{1}{r}\right) \to \infty$$

$$\to \frac{\log(c)}{\log(1/r)} \to 0 \quad \text{for any fixed constant } 0<c<\infty$$

That is, as $r \to 0$,

$\dfrac{\log(N_x(r))}{\log\left(\frac{1}{r}\right)}$ picks up the exponent $\dfrac{c}{r^n}$

and not the constant.

Computationally $\to$

It suffices to merely estimate $N_x(r)$ to within a constant.

$$\left| \frac{\log(N_x(r))}{\log(1/r)} - \frac{\log(c\,N_x(r))}{\log(1/r)} \right| \to 0 \quad \text{as } r \to 0.$$

Therefore, we might hope to define a version of dimension based on the value of $\dfrac{\log(N_x(r))}{\log\left(\frac{1}{r}\right)}$ as $r \to 0$.

Formalization

For any bounded set $X \subseteq \mathbb{R}^n$, we define

$$\dim_M(X) := \lim_{r \to 0^+} \frac{\log(N_X(r))}{\log(\frac{1}{r})}.$$

Def: $\lim_{r \to 0} f(r) = a$ iff for all $\varepsilon > 0$, $\exists \delta > 0$ s.t. for all ρ s.t. $0 < |\rho| < \delta$, $|f(\rho) - a| < \varepsilon$.

Limits are how we formalize the notion of "as $r \to 0$, $f(r) \to a$".
Limits are a big deal and form a central tool in analysis. If you have taken a calculus class, you know that integrals and derivatives are both defined in terms of limits.

We will return to limits in a bit, but for now, lets see if $\dim_M$ has nice properties.

"Nice Properties"

1. Applies to all subsets in $\mathbb{R}^n$
2. Depends on local properties
3. Invariant under a class of transformations
4. Prescribed values.

4. Prescribed Values

Need to compute $\frac{\log(N_x(r))}{\log(1/r)}$ for $X = [0,1]^n$?

But, we know it suffices to estimate $N_x(r)$ up to a constant factor. ☺

$[0,1]^n = X \qquad N_x(r) \approx \frac{1}{r^n}$ for all r small enough.

$$\to \lim_{r \to 0} \frac{\log(N_x(r))}{\log(1/r)} = \lim_{r \to 0} \frac{\log(1/r^n)}{\log(1/r)} = n. \quad ☺$$

$\dim_m$ takes the desired values on $[0,1]^n$!

3. Invariance

$f : X \to Y$ with bounded derivative

$$\frac{|f(x) - f(y)|}{|x - y|} \leq C$$

$$\to f(B_r(x)) \subset B_{Cr}(f(x))$$

$$\to N_{f(x)}(r) \leq C(n)\, N_x(r).$$

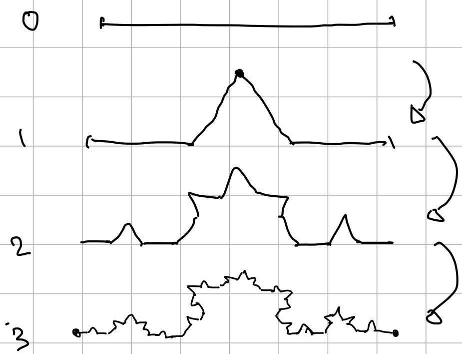
$$\text{as } r \to 0, \left| \frac{\log(N_x(r))}{\log(1/r)} - \frac{\log(N_{f(x)}(r))}{\log(1/r)} \right| \to 0$$

Thus, $\dim_m$ is invariant under transformations with bounded derivatives!

But, $\dim_m$ is <u>NOT</u> invariant under homeomorphisms.

$\boxed{\text{Ex}}$ (von Koch Snowflake)

Consider the following operation.



Now repeat this operation on each line segment.

Let $K \subset \mathbb{R}^2$ be the limit of this process

$\boxed{\text{Theory:}}$ There is a homeomorphism

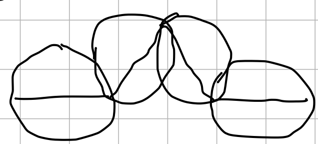
$$f: [0,1] \to K.$$

Therefore, $\dim_T(K) = 1$.

But

Q: What is $\dim_m(K)$?

If $r = \frac{3^{-n}}{2}$,



$N_K(r) \approx 4^n$

If $r \in [3^{-n}, 3^{-n+1}]$, $N_K(r) \approx 4^n$ so write $n = \frac{\log(1/r)}{\log(3)}$

$$\lim_{r \to 0} \frac{\log(N_K(r))}{\log(1/r)} = \lim_{r \to 0} \frac{\log\left(4^{\frac{\log(1/r)}{\log(3)}}\right)}{\log(1/r)}$$

$$= \lim_{r \to 0} \frac{\log(4)}{\log(3)} = \frac{\log(4)}{\log(3)} > 1.$$

- $\dim_T$ and $\dim_m$ disagree!
- and $\dim_m$ can take fractional values!
- and $\dim_m$ not invariant under homeomorphisms.

Theory: For every $0 \le s \le n$,

There is a set $X \subseteq \mathbb{R}^n$ such that

$$\dim_m(X) = s. \quad = \lim_{r \to 0} \frac{\log(N_X(r))}{\log(1/r)} \quad \text{i.e. } N_X(r) \approx \frac{1}{r^s}$$

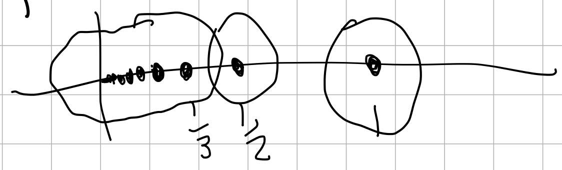
2. Depends on local properties

a. $\dim_M$ defined by balls; Nothing is more local than balls.

but, in some sense $\dim_M$ is not so local (cover all X)

On the other hand $\dim_M$ sees local properties very differently from $\dim_T$.

$\boxed{\text{Ex}}$ $X = \{ \frac{1}{n} : n \in \mathbb{N} \}$



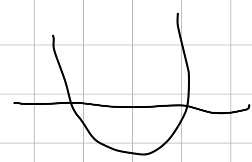
Locally, each point is isolated. $\to$ $\dim_T(X) = 0$. But $\dim_M$ sees X very differently.

- for $0 < r$ small, $\frac{1}{n}$ gets its own ball if

$$\frac{1}{n} - \frac{1}{n+1} \geq 2r$$

$$\frac{1}{n(n+1)} \geq 2r \quad \to \quad 1 \geq 2r(n^2+n)$$

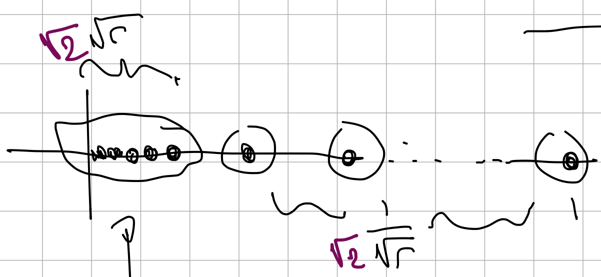
$$0 \geq n^2 + n - \frac{1}{2r}$$



$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)\left(-\frac{1}{2r}\right)}}{2}$$

$$n = \left[-1 + \sqrt{1 + 2/r}\,\right] / 2$$

for very small r, $n \approx \dfrac{-1 + \sqrt{1} + \frac{\sqrt{2}}{\sqrt{r}}}{2} = \dfrac{1}{\sqrt{2}\sqrt{r}}$



requires $\approx \frac{\sqrt{2}}{\sqrt{r}}$ balls to cover. $\to$ $N_X(r) \approx \frac{2+\sqrt{2}}{\sqrt{2}\sqrt{r}}$

$$\lim_{r \to 0} \frac{\log\left(\frac{2+\sqrt{2}}{\sqrt{2}(r^{1/2})}\right)}{\log(1/r)} = \lim_{r \to 0} \frac{\log\left(\frac{1}{r^{1/2}}\right)}{\log(1/r)} = \boxed{\frac{1}{2}}$$

This is because we built $\dim_T$ to see how sets are ~~sets are~~ connected and we built $\dim_m$ to see "how they sit in space." — and these are different.

Even though $\dim_m$ does only depend upon local properties, ... it sees things differently.

$\boxed{\text{II.}}$ applies to all subsets of $\mathbb{R}^n$.

This $\dim_m$ fails in 2 respects.

1. only can apply it meaningfully to bounded sets.

   this is actually related to some deeper issues.

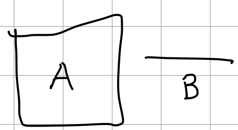
   in general, $\dim_m$ does not handle infinity very well.

   a. Infinity of unbounded sets

   b. Infinity of $\bigcup_{i=1}^{\infty} A_i$

Intuition: For $A, B \subseteq \mathbb{R}^n$, we may want

$$\dim(A \cup B) = \max\{\dim(A), \dim(B)\}$$



($\dim_T$) if A_i are closed, then

$$\dim_T\left(\bigcup_{i=1}^{\infty} A_i\right) = \sup_i \{\dim_T(A_i)\}.$$

$\to$ since single points are closed, all countable sets have $\dim_T$ zero.

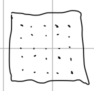
($\dim_M$) For $\dim_M$: we have that for any finite number,

$$\dim_M\left(\bigcup_{i=1}^{N} A_i\right) = \max_i \{\dim_M(A_i)\}$$

But, for countable collections

$$\dim_M\left(\bigcup_{i=1}^{\infty} A_i\right) \neq \sup_i \{\dim_M(A_i)\}.$$

$X = \{x \in [0,1]^2 : x = (\frac{a}{b}, \frac{c}{d}) \quad a,b,c,d \in \mathbb{N}\}$



$N_X(r) \approx \frac{1}{r^2} \to \dim_M(X) = 2.$

2. $\dim_M$ may **not** exist.

What does this mean?

- $\dim_M$ is defined in terms of limits.
- limits do not always exist.

$\boxed{\text{Ex}}$ let $f(x) = \sin\left(\frac{1}{x}\right)$

for all $y \in [-1,1]$, $\exists\, x_i \to 0$ s.t. $\lim_{i\to\infty} f(x_i) = y$.

$\to \lim_{r \to 0^+} f(r)$ DOES NOT EXIST.

There exist examples where

$$\lim_{r\to 0} \frac{\log(N_X(r))}{\log(\frac{1}{r})}$$ does not exist.

Even though $\dim_M$ gives us a rich, beautiful theory. there are some compelling weaknesses. Good motivation to come up with different notions of dimension.

Overcoming Infinity

Recall our exploration of $\dim_F$: we saw that the problem in comparing $\dim_F(S^1)$ and $\dim_F(\mathbb{R})$ was that we needed to consider (map) all of S^1 into $\mathbb{R}$. That is considering global properties led to difficulties, whereas localizing gave us a better notion of dimension.

For $\dim_\Lambda$, we have made the same mistake!

$N_X(r)$ depends on all of X, which is why we can only consider bounded sets $X \subseteq \mathbb{R}^n$.

One idea to overcome problems 1a. & 1b. would be to localize and give a definition more like $\dim_T$, where

$$\dim_T(X) = n \quad \text{iff} \quad \max_{p \in X} \dim_T(X, p) = n.$$

local.

$\longrightarrow$ Packing dimension $\dim_P$.

Overcoming Non-existence

It requires studying limits.

To produce a quantity that always exists:

- One idea is to study sequences $r_i \to 0$ for which $\lim_{i \to \infty} f(r_i)$ exists and consider the set of all possible limits.

→ Upper and Lower Minkowski dimension

(more locally) Assouad dimension or Lower dimension.

Assouad – Nagata dimension.

- Another idea is to modify our definition to produce a limit which is guaranteed to exist as $r \to 0$.

→ Hausdorff dimension.

Both of these ideas produce different and important notions of "dimension."

Wrap-up

"What is dimension?"

There is no one notion of dimension.

The many different notions of dimension come from formalizing different intuitions of patterns. They are all very good a capturing some aspect of that intuition, but may not capture all the behavior we care about, depending on how we want to see the world.

and, while <u>all</u> (except $\dim_{\mathcal{F}}$) agree on

a few nice model spaces,

they also all disagree on other sets (except $\dim_T$, $\dim_{LC}$).

This is a beautiful aspect of mathematics!

I hope that you have enjoyed this course.

" " exploring the question

"what is dimension?"

I know this course was a mad gallop

through topics you had never seen before,

but i hope that behind all the shiny new ideas whipping by

at break-neck speed, you can see Math as

an exploration of "mathematical ideas/objects"

based upon the creative interplay of

intuition
formalization
theory

It has been an honor to get to

talk with you this week.

Thank you so much for

your time and attention.

Theory: For $\dim_T$, all countable sets E are $\dim_T(E) = 0$. If $X \subseteq \mathbb{R}^n$, $\dim_T(X) = n$ iff X contains an open ball.

Note that for:

a. $\dim_F$ this does not hold.

$$\dim_F(\mathbb{Q}) > \dim_F(\mathbb{Q})$$

b. $\dim_T$ this is complicated. In general,

$$\dim_T(A \cup B) \leq \dim_T(A) + \dim_T(B) + 1 .$$

For all X such that $\dim_T(X) = n$

$\exists$ $n+1$ sets A_i with $\dim_T(A_i) \leq 0$

$$X = \bigcup_{i=1}^{n+1} A_i .$$