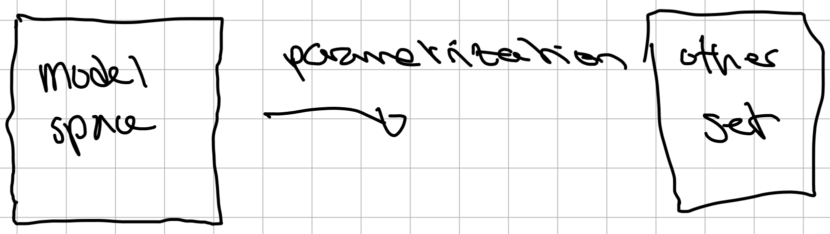
Last time, we tried to come up with a way to extend dim $\checkmark$ by parametrizing functions.



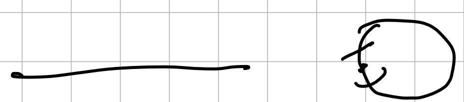
and we saw it was quite hard.

I asked you to think about how we might come up with a notion of dimension using local parametrization so that

$$\dim(C) = \dim(\mathbb{R}).$$

<u>Q:</u> what did you come up with?

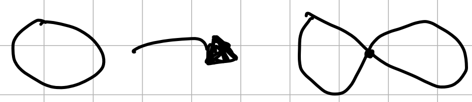
Lots of ways to do this. In <u>Manifolds</u> we say M is an n-dimensional manifold if $\forall p \in M$, $\exists r > 0$ s.t. $B_r(p) \cap M \sim \mathbb{R}^n$.



When physicists say that they use models which use $4, 5, \ldots, 10, 11$ dimensions, THIS is the definition they are using.

__But__ this definition and __any__ definition of dimension based on parametrization will face major problems.

- For any list of model spaces, we can produce a space that cannot be parametrized.

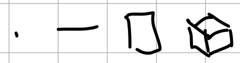
$\boxed{\text{Ex}}$ 

If we want a definition of dimension that can apply to __all__ subsets in $\mathbb{R}^n$, parametrization is __doomed to failure__.

We need a __new__ approach.

To help guide us, lets consider what kinds of properties we want dimension to satisfy.

$\boxed{1.}$ Applies to all subsets of $\mathbb{R}^n$.

$\boxed{2.}$ Takes prescribed values. 

$\boxed{3.}$ Should only depend on local properties.

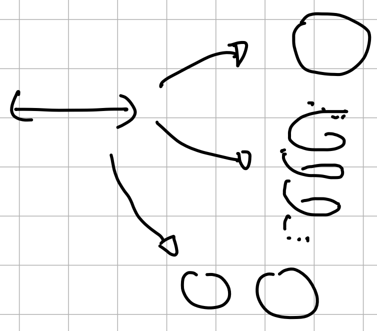
$\boxed{4.}$ For nice functions f,

$$\dim(f(X)) = \dim(X).$$

The ~~fancy~~ word for this is that we expect dimension to be __invariant__ under some (but not all) transformations.

When we were thinking of f as a parametrization, our definition of dimension applied to any set which looks everywhere locally like $\mathbb{R}^n$.

$\boxed{n=1}$



But, as mentioned, we were always able to find some annoying exception which we wanted to be dimension one. ∞.

By changing our thinking to "dimension as invariant under a class of transformations", we are no longer trying to map $\longleftrightarrow$ to everything we think should be 1-dim. Instead, we are trying to study properties of sets that are preserved by these transformations.

Since sets which may not be mutually parametrizable may still share these properties, we are, in a sense, trying to find general properties which are the secret sources of our intuition, rather than trying to base our definition on $\mathbb{R}^n$ and extending it outwards.

So, ... This is the hope.

to try to make it happen, lets

a. Pick a class of transformations

b. Try to find some properties which are preserved.

c. See if a pattern emerges.

a. Choose a class of functions.

a. bijections $\longrightarrow$ cardinality (does not agree w/ Intuition)

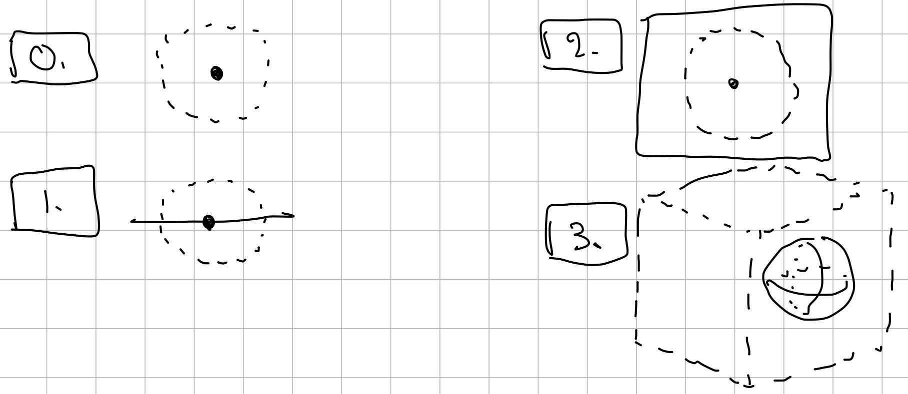
b. homeomorphisms $\longrightarrow$ Topology

c. Functions with Bounded derivatives $\longrightarrow$ metric theory

Since we have been working with homeomorphisms, let's say (for now) we want a definition of dimension which is invariant under homeomorphisms.

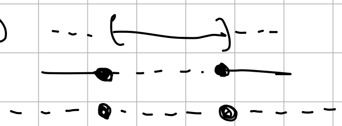
b. Find some properties. want "local" properties.

By "local" we mean, "look at small balls."



Let's describe everything we see!

1. obviously, there is the ball $B_r(p)$
2. There is also $B_r(p)^c$

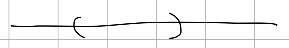


3. There is also some special points. The boundary

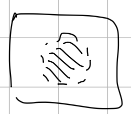
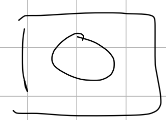
- Universe = point

  ∂ of point $= \emptyset$.

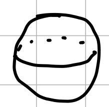
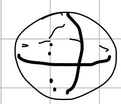
- Universe = Line

 $\partial B_r(p) =$ 

- Universe = Plane

 Boundary is 

- Universe is $\mathbb{R}^3$

 Boundary is  sphere.

a pattern emerges!

Boundary of 3-dim thing $\to$ 2 dim thing

Boundary of 2 dim thing $\to$ 1 dim thing

1 dim $\to$ 0 dim

0 dim $\to \emptyset$.

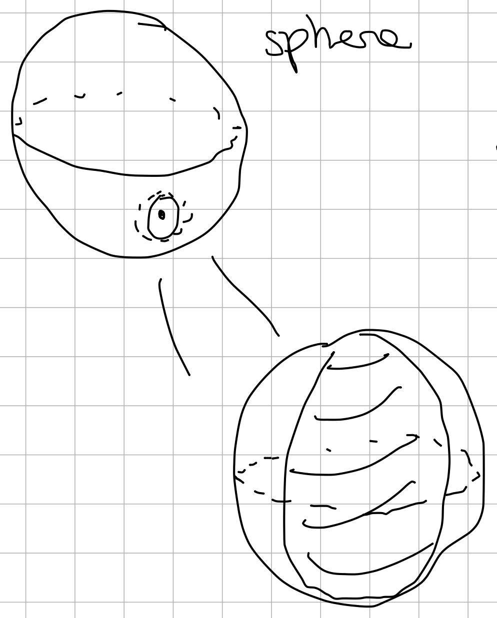
This was first expressed by Poincare in 1912 just before he died. This is our intuition.

Lets make it rigorous.

<u>Def</u>: Let $U \subseteq \mathbb{R}^n$. We define the boundary of U

$$\partial U = \{ x \in \mathbb{R}^n \text{ s.t. } \forall r > 0 \quad B_r(x) \cap U \neq \emptyset, \quad B_r(x) \cap (U^c) \neq \emptyset \}$$

→ Lets test it!



1. pick a point $p \in S^2$
2. Consider $B_r(p) \cap S^2$
3. Look at $\partial(B_r(p) \cap S^2)$

Problem:

$$\partial(B_r(p) \cap S^2) = \overline{B_r(p)} \cap S^2 \quad !!$$

(2-dim)

The problem is that our universe is $\mathbb{R}^3$ not S^2.

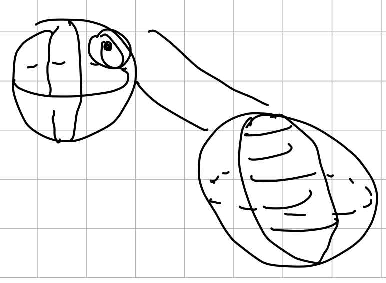
We need a notion of boundary that only sees the sets we care about.

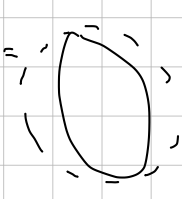
Def: Let $A \subseteq U \subseteq \mathbb{R}^n$. We define the

boundary of A relative to U

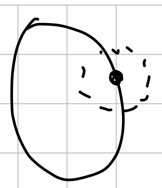
$$\partial^U A = \{x \in U : \forall r > 0 \;\; B_r(x) \cap A \neq \emptyset, \;\; B_r(x) \cap (U \setminus A) \neq \emptyset\}$$

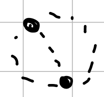
Back to the Example:



$\partial^{S^2}(B_r(p) \cap S^2) =$ 

which is 1-dimensional!!



$\partial^{O}(B_r(p) \cap O) =$ 

0-dim.

$\partial^{\cdot\cdot}(B_r(p) \cap \cdot\cdot) = \emptyset.$

$\boxed{\text{Big Theory Fact}}$: Boundaries are preserved by homeomorphisms
this is related to preserving connected sets.

<u>Def</u>: Let $X \subset \mathbb{R}^n$. Let $0 \leq m \leq n$ in $\mathbb{N}$.

1. $\dim_T(\emptyset) = -1$

2. for $p \in X$, we say

$$\dim_T(X, p) \leq m$$

<u>iff</u> $\forall\, U \subseteq \mathbb{R}^n$ open,

$\exists\, V$ nbhd of p

$p \in V \subseteq U$ such that

$\partial^X V \to$ $\dim_T(X \cap \partial V) \leq m - 1$

3. for $p \in X$, we say $\dim_T(X,p) = m$

<u>iff</u> $\dim_T(X,p) \leq m$

but <u>not</u> $\dim_T(X,p) \leq m-1$.

4. We say $\dim_T(X) \leq m$

<u>iff</u> $\dim_T(X,p) \leq m \ \forall p \in X$

5. We say

$$\dim_T(X) = m$$

iff

$$\dim_T(X) \le m \text{ but not}$$

$$\dim_T(X) \le m-1.$$

Remark:

The empty set is -1 dimensional!

in many ways, this is done just to follow the pattern of boundaries being one less dimension. But it is pretty wild!

$\boxed{\text{OK}}$ Now that we have a definition,

Q: Is it invariant under homeomorphisms?

Yes. Because Boundaries are preserved.

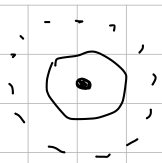
Q: Does it depend upon local properties?

Yes.

Q: Does it give us the prescribed values?

$\dim_T(\mathbb{R}^n) = n$?

Let's check!

$\boxed{0.}$ 

$\{p\}\cap\partial V=\emptyset$ for all V.

$\to \dim_T(\{p\}\cap\partial V)=-1 \quad \forall V$

$\to \dim_T(\{p\})\le 0$

and $\dim_T(\{p\})\ne -1$.

$\to \dim_T(\{p\})=0$.

$\boxed{1.}$ 

We can choose $V=B_r(p)$

$\partial V=$ two points $\to$ 0-dimensional.

$\to \dim_T(\mathbb{R})\le 1$

Is it possible $\dim_T(\mathbb{R})\le 0$?

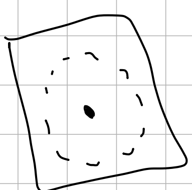
$\to$ for all $p\in\mathbb{R}$, $\exists$ V open set s.t. $p\in V$, $V\subseteq B_r(p)$ & $\partial V=\emptyset$?

If such a V exists, we can reduce to

convex hull.

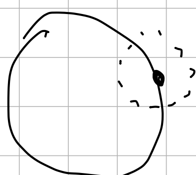
which implies that $\mathbb{R}$ is disconnected. But $\mathbb{R}$ is connected.

Therefore, $\dim_T(\mathbb{R})=1$.

$\boxed{2.}$ 

$U=B_{2r}(p)$, choose $V=B_r(p)$.

$\partial V_1=$ ○ circle.



let $V_2=B_{r'}(p')$

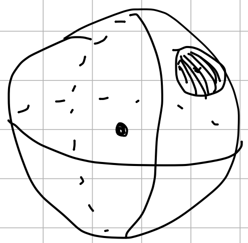
$\partial V_2\cap\partial V_1\cap\mathbb{R}^2=$ two points.

$=$ 0-dim

$\to \partial V_1 =$ circle has

$$\dim_T(\partial V_1) \le 1$$

$$\longrightarrow \dim_T(\mathbb{R}^2) \le 2$$

$\boxed{13.}$  $U = B_{2r}(p)$, choose $V = B_r(p)$

$\partial V = S^2$

Sphere is locally homeomorphic to $\mathbb{R}^2$.

$$\to \dim_T(\partial V) \le 2$$

$$\longrightarrow \dim_T(\mathbb{R}^3) \le 3.$$

- In general, it is not so hard to show

$$\dim_T(\mathbb{R}^n) \le n.$$

But proving $\dim_T(\mathbb{R}^n) = n$ is harder.

If we want to approach it directly,

we need to show that for all bounded, open neighborhoods $V \subseteq \mathbb{R}^n$

$$\dim_T(\partial V) \geq n-2.$$

This is possible for $n = 0, 1$

but no known direct proof for $n \geq 2$.

Instead, to prove $\dim_T(\mathbb{R}^n) = n$ we could show that $\mathbb{R}^n$ is not $(n-1)$-dimensional because $\mathbb{R}^n$ possesses some property <u>all</u> $(n-1)$-dimensional sets lack.

Take-Home Question

<u>Q:</u> What properties does $\mathbb{R}^n$ have that $\mathbb{R}^{n-1}$ does not?