WHAT IS DIMENSION? DAY 4

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1. Day 4 Notes

Definition 1.1. Let $U \subset \mathbb{R}^n$ and let $X, E_1, E_2 \subset U$ be disjoint subsets. Assume that E_1, E_2 are non-empty. We say that X *separates* E_1 and E_2 in U if anf only if there exist two open sets W_1, W_2 such that the following properties hold.

(1) $W_1 \cap W_2 = \emptyset$.
(2) $U \setminus X \subset W_1 \cap W_2$.
(3) $E_1 \subset W_1$ and $E_2 \subset W_2$.

Theorem 1.2. *Let $\{C_i, C_i'\}_{i=1}^n$ be the opposite faces of the n-cube $[0,1]^n \subset \mathbb{R}^n$. Let K_i be closed sets separating C_i, C_i' in $[0,1]^n$. Then*

$$\bigcap_{i=1}^n K_i \neq \emptyset.$$

Theorem 1.3. *If X has $\dim_T(X) \leq n-1$, and $\{C_i, C_i'\}_{i=1}^n$ are n pairs of closed sets in X such that $C_i \cap C_i' = \emptyset$ then there exists a collection $\{B_i\}_{i=1}^n$ such that B_i separates C_i, C_i' in X and*

$$\bigcap_{i=1}^n B_i = \emptyset.$$

Definition 1.4. Let $E \subset \mathbb{R}^n$. A *closed (open) cover* of E is a collection of closed (open) balls $\{B_{r_i}(x_i)\}$ such that

$$E \subset \bigcup_i B_{r_i}(x_i).$$

For each $x \in E$, we define the *covering index at x* of a closed cover $\{K_i\}_i$ of $E \subset \mathbb{R}^n$ to be the number of K_i such that $x \in K_i$.

We define the *covering index* of $\{K_i\}_i$ to be the maximum of the covering indices at $x \in E$.

Definition 1.5. For $X \subset \mathbb{R}^n$ we say that

$$\dim_{LC}(X) = n$$

if and only if every closed cover has a refinement with cover index $\leq n+1$.

2. Problem Set # 4

2.1. Computation Problems.

Definition 2.1. Consider the following set-up. Let $E \subset \mathbb{R}^n$ be a bounded set. This means there is some $0 < R$ such that $E \subset B_R(0)$. We define the *counting number* of E at scale r, written $N_E(r)$, to be the smallest number of open balls $B_r(x)$ required to cover E.

(1) Estimate $N_{[0,1]}(r)$ for all $0 < r < \infty$.

Sean's note: We may easily estimate

$$N_{[0,1]}(r) \cong \begin{cases} 1 & r > 1/2 \\ 1/r & 0 < r \leq 1/2. \end{cases}$$

Ask students if this changes if we thing of $[0,1] \hookrightarrow \mathbb{R}^3$?

(2) Estimate $N_{[0,1]^2}(r)$ for all $0 < r < \infty$.

Sean's note: We may easily estimate

$$N_{[0,1]^2}(r) \cong \begin{cases} 1 & r > \sqrt{2}/2 \\ \frac{1}{r^2} & 0 < r \leq \sqrt{2}/2. \end{cases}$$

(3) Estimate $N_{[0,1]^3}(r)$ for all $0 < r < \infty$.

Sean's note: We may estimate

$$N_{[0,1]^3}(r) \cong \begin{cases} 1 & r > \sqrt{3}/2 \\ \frac{1}{r^3} & 0 < r \leq \sqrt{3}/2. \end{cases}$$

(4) Let $E \subset \mathbb{R}^2$ be the rectangle $E = [0,100] \times [0,1]$. Estimate $N_E(r)$ for all $0 < r < \infty$.

Sean's note: We may easily estimate

$$N_E(r) \cong \begin{cases} 1 & r > 101/2 \\ \frac{1}{r} & r \in [\sqrt{2}/2, 101/2 \\ \frac{1}{r^2} & 0 < r \leq \sqrt{2}/2. \end{cases}$$

(5) Let $E \subset \mathbb{R}^3$ be the block $E = [0,1] \times [0,100] \times [0,1000]$. Estimate $N_E(r)$ for all $0 < r < \infty$.

Sean's note: We may estimate

$$N_E(r) \cong \begin{cases} 1 & r > 1101/2 \\ \frac{1}{r} & r \in [101/2, 1101/2] \\ \frac{1}{r^2} & r \in [\sqrt{2}/2, 101/2] \\ \frac{1}{r^3} & 0 < r\sqrt{2}/2. \end{cases}$$

The exact cut-offs for the ranges of r are NOT important.

2.2. **Exploration Problems.** The Big Question is:

What is $\dim_{LC}([0,1]^n)$?

To help guide you in this exploration, you may wish to consider the following questions.

1. (Get Intuition) Since we only care about coverings by balls with small radii, draw an example of a covering of $[0,1]^2$ by balls with radius $\sim 1/4$.
   a. Staring at this picture, is there any way that you can begin to use the language of separating sets to describe parts of the picture? How might you find sets which separate the faces of the square $[0,1]^2$?
2. Is there a big theorem that might help? If so, does the theorem finish the problem?

Sean's note: This problem involves many steps. Please help guide the students through it using questions and suggestions.

1. Note that the union of all the balls that intersect one face C_i of the cube do not intersect the opposite face $C_i^{'}$. This means that the boundary of the union of these balls is a set which separates C_i, C_i' in the cube $[0,1]^n$.
2. We have Theorem 1.2, which says that the intersection of these n separating sets must have a non-empty intersection.
3. This does NOT solve the problem. It only gives a covering index of n. We still need to show that we get $n+1$.
4. The key is that we can choose this intersection point to be on the boundary. This means there must be another ball which covers it, giving $n+1$. This is easy to see inductively.
5. This gives $\dim_{LC}(\mathbb{R}^n) = n$.

There is no need for students to solve this problem, and I do NOT care about rigor. I DO want them to explore the ideas and get them basically correct.

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